COHERENT AND STOCHASTIC PARTICLE MOTION IN A UNIFORM MAGNETIC FIELD AND OBLIQUE ELECTROSTATIC WAVES

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Supported by NSF/DOE Grant DE-FG02-99ER54555,
NSF Grant ATM 98-06328, DOE Grant DE-FG02-91ER54109

2002 Sherwood Conference
Rochester, NY
Poster 2C48
23 April 2002

ABSTRACT

Stochastic motion of particles in the presence of a background magnetic field and electrostatic waves is of interest in both laboratory and space plasmas. Ion heating in fusion experiments can be achieved by a single wave driving particles into chaotic dynamics. Wave mechanisms are also believed to account for the energization of ions from the ionosphere to the magnetosphere. In particular, it has recently been shown that two perpendicular waves may explain the high-energy tail of H^+ and O^+ distributions in the upper ionosphere [1].

In a uniform magnetic field $\vec{B}_0$, one perpendicular wave causes particles within a range of perpendicular energies to move stochastically when the wave amplitude exceeds a threshold [2,3]. Two perpendicular waves whose frequencies differ by an integer multiple of the cyclotron frequency ($\omega_1 - \omega_2 = N\Omega_c$) can coherently accelerate particles from low energies into the one-wave chaotic regime. We show that oblique waves can also produce coherent acceleration, provided their parallel wavenumbers are sufficiently close. The resonance condition now applies to the Doppler-shifted wave frequencies: $(\omega_1 - k_{1z}v_{z0}) - (\omega_2 - k_{2z}v_{z0}) = N\Omega_c$.

We also consider the possibility of enhanced ion heating with two oblique waves as opposed to one oblique wave. Each wave produces a chain of primary islands at $k_{iz}v_z - \omega_i = N\Omega_c$, and stochasticity occurs when islands overlap [4,5]. Both one and two waves produce a rapid initial energization of particles in the stochastic region followed by slower, long-term heating. For two waves, the heating is found numerically to be more effective, and the stochastic region is extended to lower v_z.

Outline

Coherent Acceleration

- 1 perpendicular wave: stochastic region for $k_\perp \rho_i \equiv r \sim \nu \equiv \omega/\Omega_c$
- 2 perpendicular waves, $\omega_1 - \omega_2 = N\Omega_c$: coherent acceleration of low-energy particles into stochastic region
- Finite $k_z : \omega_1 - \omega_2 - (k_{1z} - k_{2z})v_{z0} = N\Omega_c$
  - Lie perturbation analysis of coherent motion
  - When does coherent acceleration occur?
  - Coherent change in v_z

Stochastic Heating by Oblique Waves

- 1 oblique wave: Overlap of islands leads to stochastic region
- 2 oblique waves: 2 chains of islands; energization
- Heating primarily in the parallel direction
- Faster and stronger heating, wider stochastic region with 2 waves

Particle Equations of Motion

• Particle moving in uniform background $\vec{B}_0 = B_0\hat{z}$ and electrostatic waves:

$$\ddot{\vec{x}} = \sum_{i=1}^{2} \epsilon_i \vec{k}_i \sin(\vec{k}_i \cdot \vec{x} - \nu_i t) + \vec{v} \times \hat{z}$$

$$\epsilon_i \equiv \frac{q\Phi_i}{m\Omega_c^2}, \qquad \nu_i = \frac{\omega_i}{\Omega_c}, \qquad \vec{k}_i = (k_{ix}, 0, k_{iz})$$

Time scaled to $\Omega_c \equiv qB_0/m$, length to typical k^{-1}

• Hamiltonian formulation:

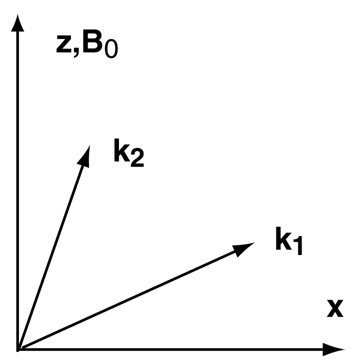
$$H_w[\vec{q}, \vec{p}, t] = \frac{1}{2}(\vec{p} - \vec{A})^2 + \sum_i \epsilon_i \cos(\vec{k}_i \cdot \vec{x} - \nu_i t)$$

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{A} = B_0 x \hat{y}$$

• Guiding Center variables:

$$H = P_\phi + \frac{1}{2}v_z^2 + \sum \epsilon_i \cos(k_{ix} r \sin\phi + k_{iz} z - \nu_i t)$$

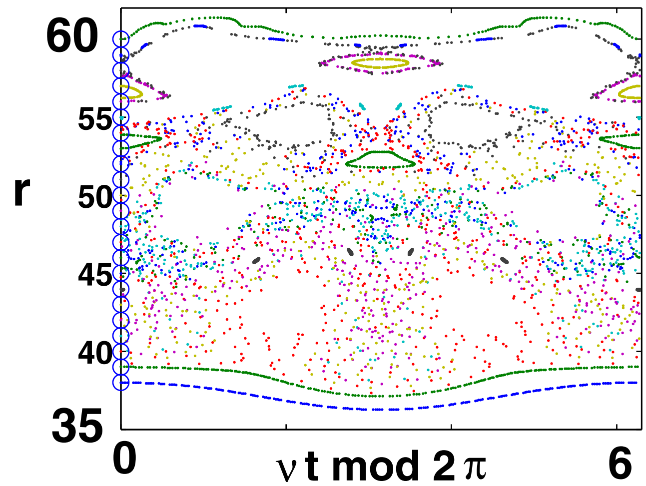
$$\phi = \arctan\frac{x}{v_x} = \text{gyrophase}, \qquad r = \sqrt{2P_\phi} = \sqrt{x^2 + v_x^2} = \text{Larmor radius}$$



One Perpendicular Wave[2,3]

Stochastic region: $\nu - \sqrt{\epsilon} < r < (2/\pi)^{1/3}(4\epsilon\nu)^{2/3}$

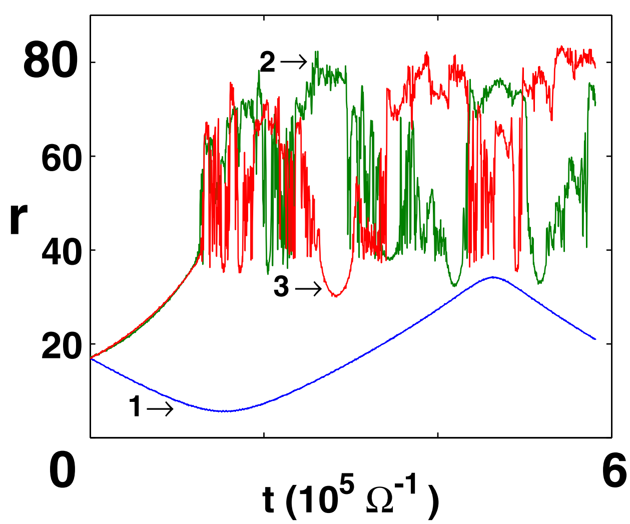
$\nu = 40.47, \epsilon = 3.2$ Surface of Section (ϕ=0)



Two Perpendicular Waves[1]

Coherent Acceleration: $\nu_1, \nu_2 \notin \mathcal{Z}, \quad \text{but} \quad N \equiv \nu_1 - \nu_2 \in \mathcal{Z}$

$\nu_1 = 40.47, \nu_2 = 39.47, \epsilon_1 = \epsilon_2 = 3.2, k_{1x} = k_{2x} = 1$



Analytic Treatment

- Average out fast, incoherent motion to get equation for slow, coherent motion
- Use Lie transform methods to find Hamiltonian for coherent motion
- Bounds of motion in r and v_z
- Understanding of phase dependence of energization

Effects of finite k_z

- Coherent change in v_z : $\Delta v_z = \frac{k_{1z}-k_{2z}}{2N}(r^2 - r_0^2)$
- When does coherent r acceleration remain? Answer: small $(k_{1z} - k_{2z})$
- Large k_z with small $(k_{1z} - k_{2z})$ gives acceleration, but incoherent fluctuations in $v_z \gg$ coherent change in v_z

Lie Perturbation Theory[6] for Two Oblique Waves

- Lie generating function w gives new coordinates $\bar{x}$ w/ Hamiltonian $\bar{H}$:

$$\frac{d\bar{x}}{d\epsilon} = [\bar{x}, w]_{\bar{x}}, \qquad \bar{x}(\epsilon = 0) = x \qquad [f, g]_x = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

- w,$\bar{H}$ can be found perturbatively: $w = \sum \epsilon^i w_i$, ...

$$\begin{aligned} D_0(f(x)) &\equiv \partial_t f + [f, H_0]_x \\ D_0 w_1 &= \bar{H}_1 - H_1 \\ D_0 w_2 &= 2(\bar{H}_2 - H_2) - [w_1, \bar{H}_1 + H_1]_x \end{aligned}$$

For $\bar{x}$ to describe coherent motion, choose $\bar{H}_i$ to kill resonant terms on RHS of equation for w_i.

$$\begin{aligned} H &= H_0 + \epsilon H_1 \\ H_0 &= P_\phi + \tfrac{1}{2} v_z^2 \\ H_1 &= \epsilon_i \cos(k_{ix} r \sin\phi + k_{iz} z - \nu_i t) \\ &= \sum_{m=-\infty}^{\infty} \epsilon_i J_{i,m} \cos(m\phi + k_{iz} z - \nu_i t), \qquad J_{i,m} \equiv J_m(k_{ix} r) \\ \bar{H} &= H_0 + \epsilon \bar{H}_1 + \epsilon^2 \bar{H}_2 \end{aligned}$$

- Unperturbed motion:

$$P_{\phi_u} = P_{\phi 0} \qquad v_{zu} = v_{z0} \qquad \phi_u = t + \phi_0 \qquad z_u = v_{z0} t + z_0$$

Second-Order Perturbation Theory

- H_1 contains no resonant terms, so choose $\bar{H}_1 = 0$. Gives w_1:

$$w_1 = -\sum_{il} \frac{\epsilon_i J_{i,l}}{l + k_{iz} v_z - \nu_i} \sin[l\phi + k_{iz} z - \nu_i t]$$

$$D_0 w_2 = (\partial_t + \partial_\phi + v_z \partial_z) w_2 = 2\bar{H}_2 - [w_1, H_1]$$

- $[w_1, H_1]$ contains resonant terms from

$$\cos[(l-m)\phi + (k_{iz} - k_{jz})z - (\nu_i - \nu_j)t] \rightarrow \cos[(l - m + (k_{iz} - k_{jz})v_{z0} - \nu_i + \nu_j)t + c]$$

Set $\bar{H}_2$ equal to resonant terms:

$$\bar{H}_2 = S_{10}[P_\phi, v_z] + S_2[P_\phi, v_z] \cos[N(\phi - t) + (k_{1z} - k_{2z})z]$$

Resonance Condition: $\boxed{\nu_1 - \nu_2 - (k_{1z} - k_{2z})v_{z0} \in \mathcal{Z}}$

Change variables to I, ψ using the generating function $F_2 = I(\phi - t)$:

$$\psi = \frac{\partial F_2}{\partial I} = \phi - t \qquad P_\phi = \frac{\partial F_2}{\partial \phi} = I = \frac{r^2}{2} \qquad \tilde{H} = \bar{H} + \frac{\partial F_2}{\partial t} = \bar{H} - I$$

$$\boxed{\tilde{H}[P_\phi, v_z, \psi, z] = S_1[P_\phi, v_z] + S_2[P_\phi, v_z] \cos[N\psi + (k_{1z} - k_{2z})z]}$$

$$\rightarrow \boxed{v_z[r] = v_{z0} + \frac{k_{1z} - k_{2z}}{N} \frac{r^2 - r_0^2}{2}}$$

- Bounds on $\tilde{H}$: Particle confined in r to stay between $H_\pm$

$$\boxed{H_- \leq \tilde{H} \leq H_+ \qquad \text{where} \quad H_\pm[r] \equiv S_1[r] \pm |S_2[r]|}$$

Expressions for S_1, S_2

$$S_1 = \frac{1}{2}v_z^2 + S_{1x} + S_{1z}$$

*** $\frac{1}{2}v_z^2$ absent for perp. waves!

$$S_{1x} = -\frac{k_{ix}\epsilon_i^2}{2r}\frac{l}{l + k_{iz}v_z - \nu_i}J_{i,l}J'_{i,l}$$

$$S_{1z} = \frac{1}{4}\frac{k_{iz}^2\epsilon_i^2}{l + k_{iz}v_z - \nu_i}J_{i,l}^2$$

$$S_2 = S_{2x} + S_{2z}$$

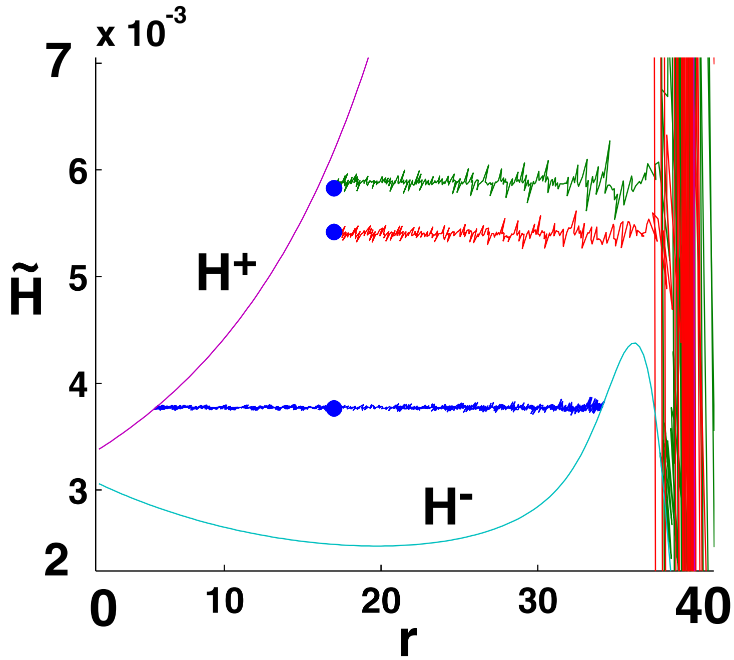
$$S_{2x} = -\frac{\epsilon_1\epsilon_2}{4r}\left(\frac{1}{l + k_{1z}v_z - \nu_1} + \frac{1}{l + k_{2z}v_z - \nu_1}\right) * \left(k_{1x}(l - N)J'_{1,l}J_{2,l-N} + k_{2x}lJ'_{2,l-N}J_{1,l}\right)$$

$$S_{2z} = \frac{k_{1z}k_{2z}\epsilon_1\epsilon_2}{4}\left(\frac{1}{(l + k_{1z}v_z - \nu_1)^2} + \frac{1}{(l + k_{2z}v_z - \nu_1)^2}\right)J_{1,l}J_{2,l-N}$$

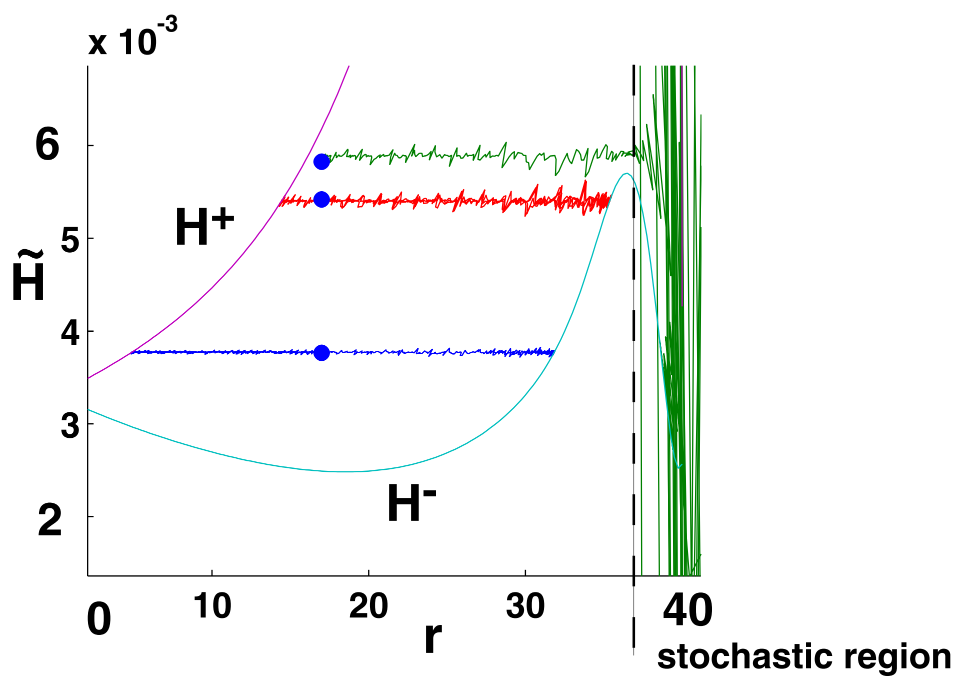
$$J'_{i,l} = \frac{dJ_l(k_{ix}r)}{d(k_{ix}r)}$$

Hamiltonian Bounds Explain r Motion, ϕ Dependence

Perpendicular Waves: $\nu_1 = 40.47, \nu_2 = 39.47, \epsilon_1 = \epsilon_2 = 3.2$

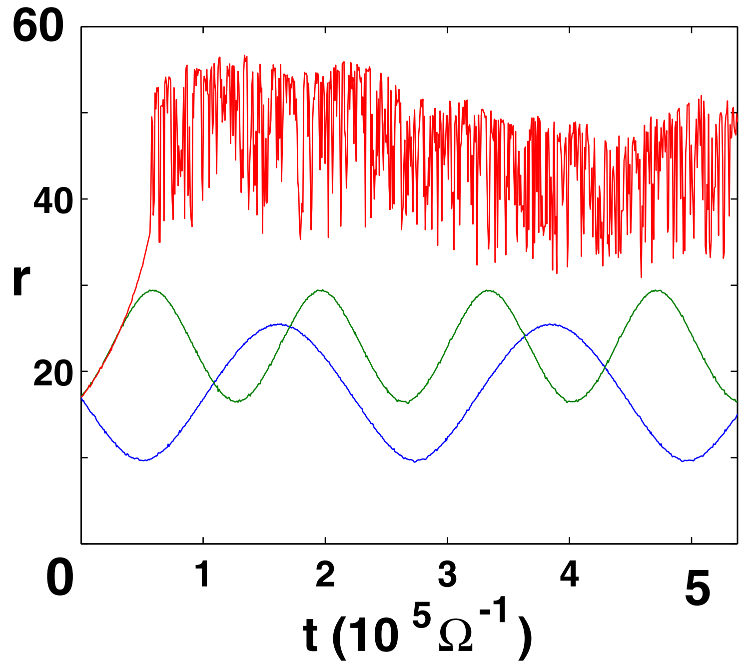


Oblique Waves: $k_{1z} = 10^{-4}, k_{2z} = 0$: Coherent Acceleration Modified

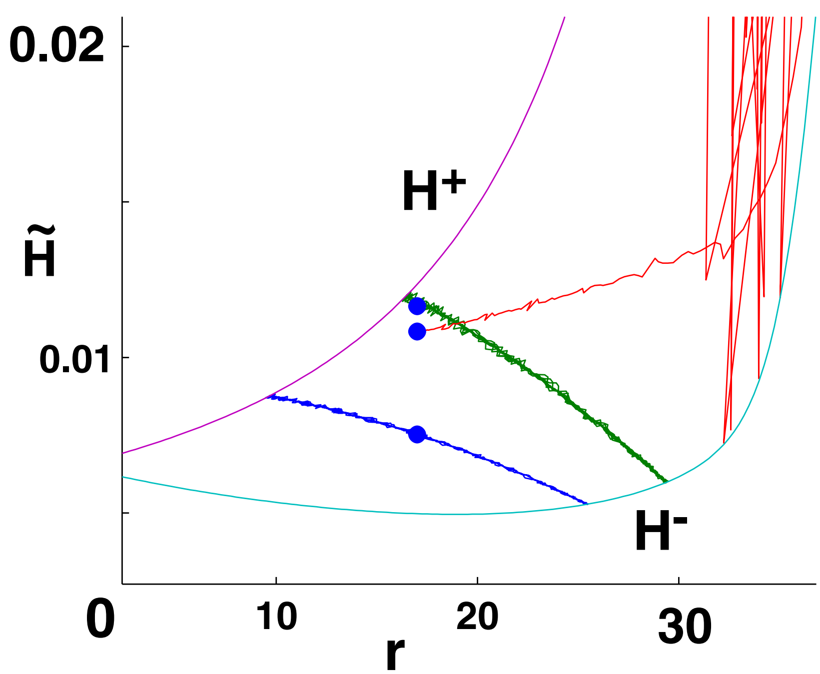


$k_{1z} = 1.0001, k_{2z} = 1$: $\tilde{H}$ Not Constant?

r vs. t

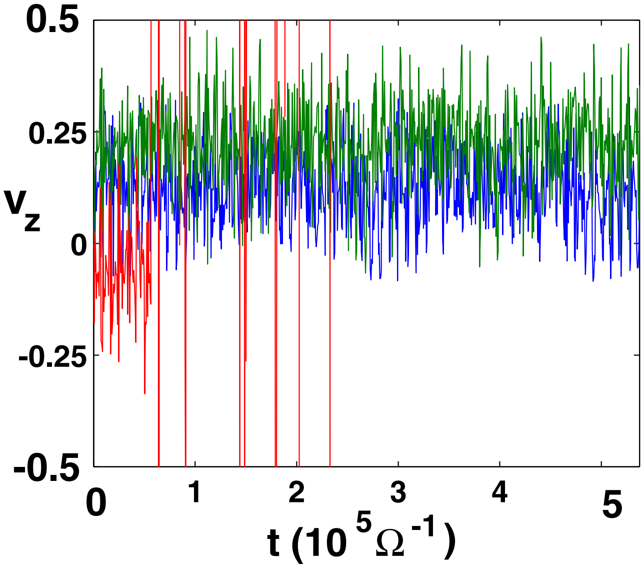


$\tilde{H}$ vs. r



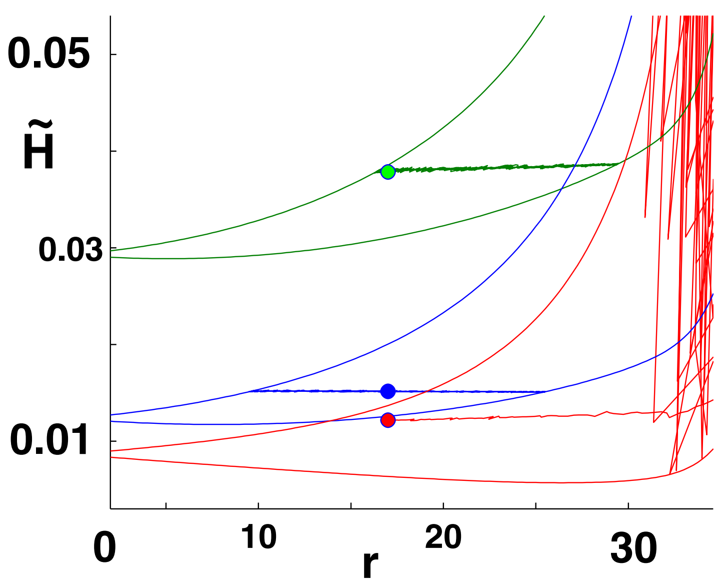
$k_{1z} = 1.0001, k_{2z} = 1$: v_z fluctuations

v_z vs. t



- v_z fluctuations $\gg$ coherent motion. Add $\langle v_z \rangle$ to v_{z0} for $H_\pm$.

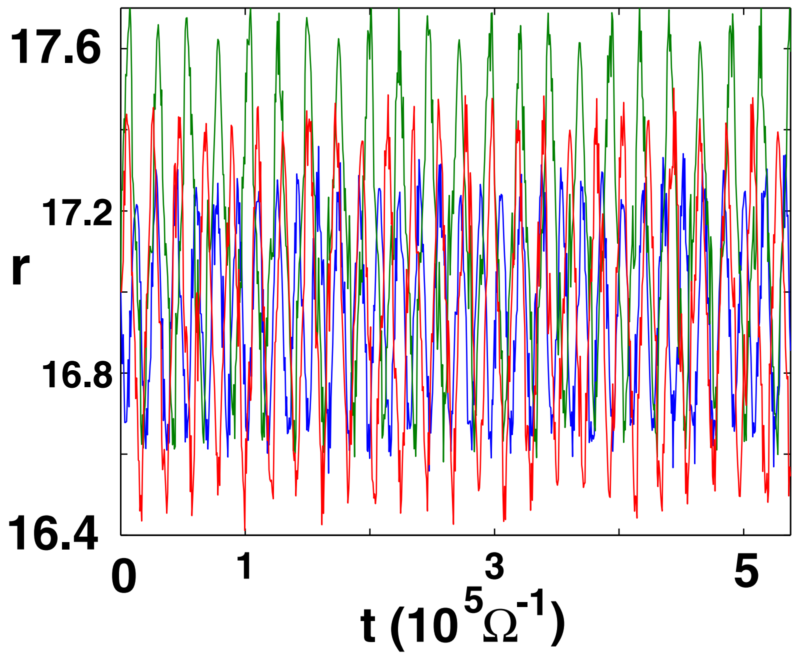
$\tilde{H}$ with $v_z(0) = \langle v_z \rangle$ vs. t



- Separate $H_\pm$ for each set of initial conditions.

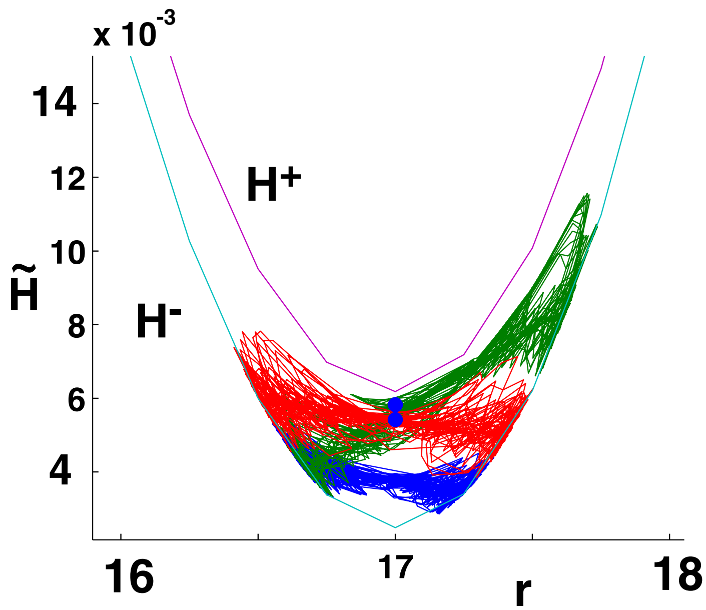
$k_{1z} = 0.01, k_{2z} = 0$: Coherent Acceleration Suppressed

r vs. t



- Larger $v_z^2/2$ term brings H_+, H_- closer.

$\tilde{H}$ vs. r



One Oblique Wave[4]

Transform to wave frame ($v_z \to v_z - \nu/k_z$):

$$H = \frac{1}{2}(r^2 + v_z^2) + \sum_l \epsilon J_l(k_x r)\cos(l\phi + k_z z)$$

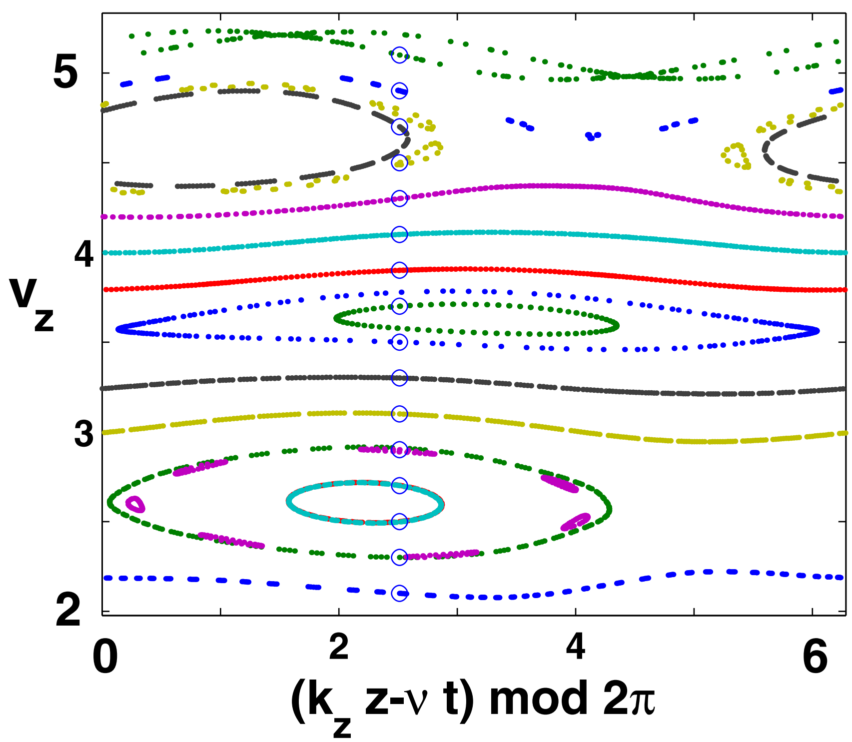
- $\epsilon \ll 1$: Series of resonances at $v_z - \nu/k_z = L \in \mathcal{Z}$. H near each resonance $\approx$ nonlinear oscillator:

$$H_L \approx E + \epsilon J_L(k_x r)\cos(L\phi + k_z z)$$

Island half-width in v_z : $w_L \equiv 2\sqrt{\epsilon J_L(k_x r)}$

Resonance Overlap Criterion: $w_L + w_{L\pm 1} > 1/k_z$

$\phi = 0.3\pi$ Surface of Section: $\epsilon = 0.06, \nu = 3.6, k_x = k_z = 1, r_0 = 2.24$



Two Oblique Waves[5]

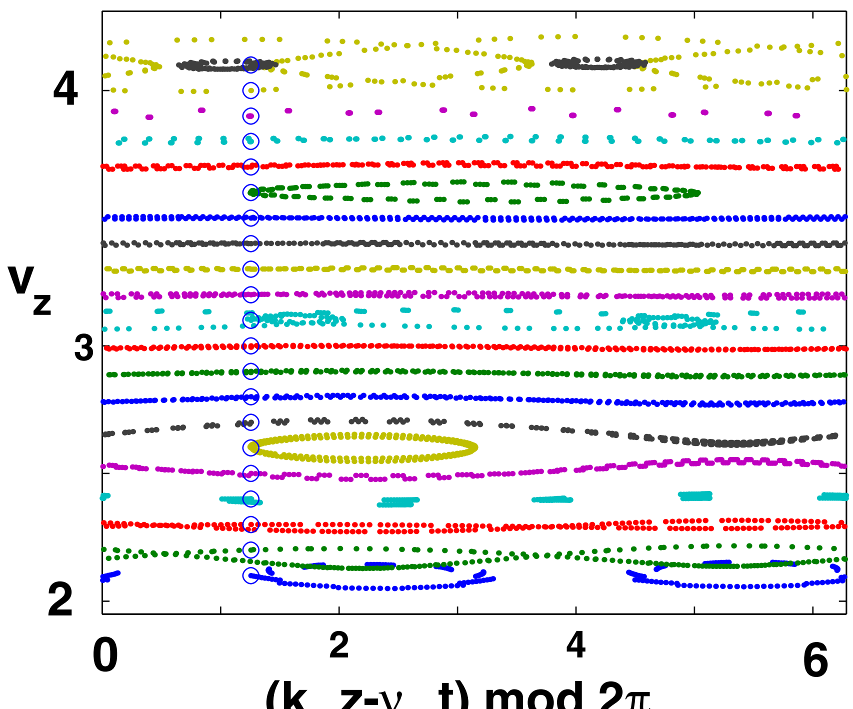
- Cannot generically eliminate time from both waves:

$$H = \frac{1}{2}(r^2+v_z^2)+\epsilon_1\cos(k_x r\sin\phi+k_z z)+\epsilon_2\cos(k_x r\sin\phi+k_z z-(\nu_2-\nu_1)t)$$

- H not constant, cannot bound kinetic energy as in one-wave case.

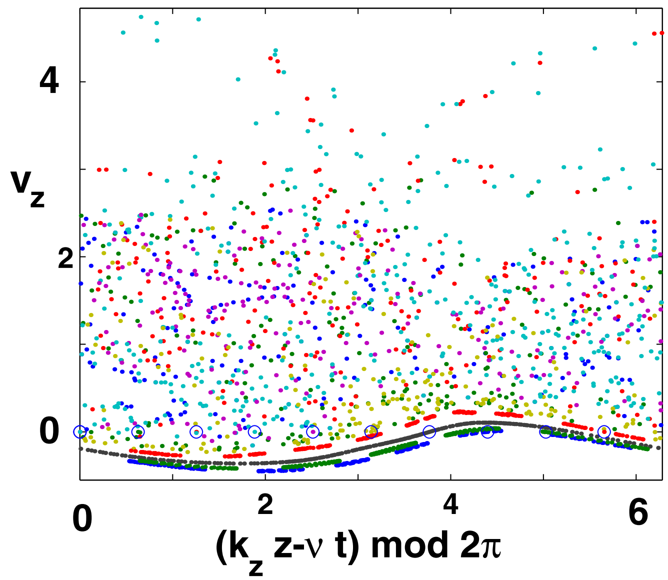
- $\epsilon \ll 1$: Each wave produces its own chain of islands

$\phi = 0.3\pi$ SoS; $\epsilon = 0.005, \nu = 3.6, 3.1, k_x = k_z = 1, r_0 = 1.4$



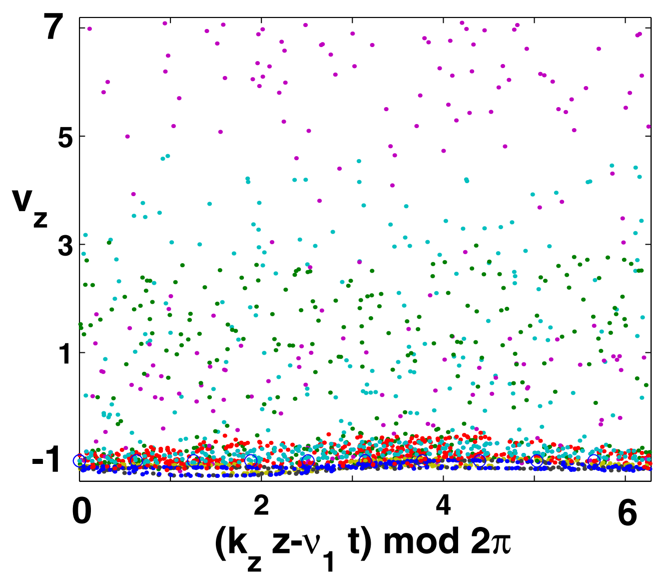
One Large-Amplitude Wave

$\epsilon = 0.4, k_x = k_z = 1, \nu = 3.6, v_{z0} = 0, r_0 = 2.24$



Two Large-Amplitude Waves

$\epsilon = 0.2, k_x = k_z = 1, \nu = 3.6, 3.1, v_{z0} = -1, r_0 = 2.24$

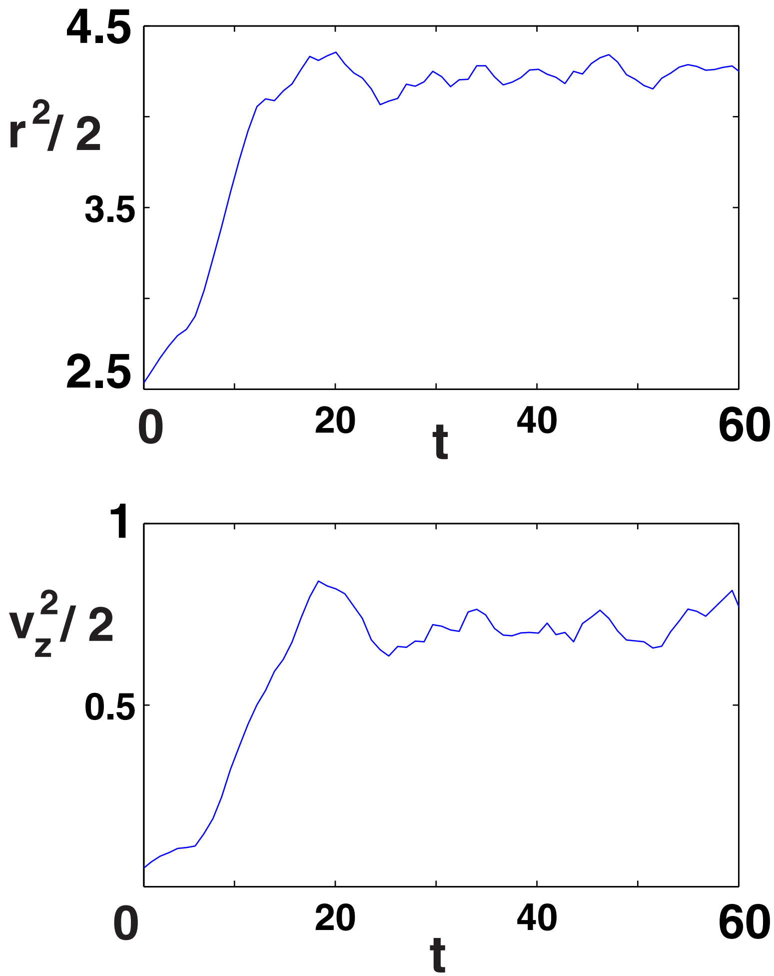


- 2 waves: Larger stochastic region in v_z with half the wave energy

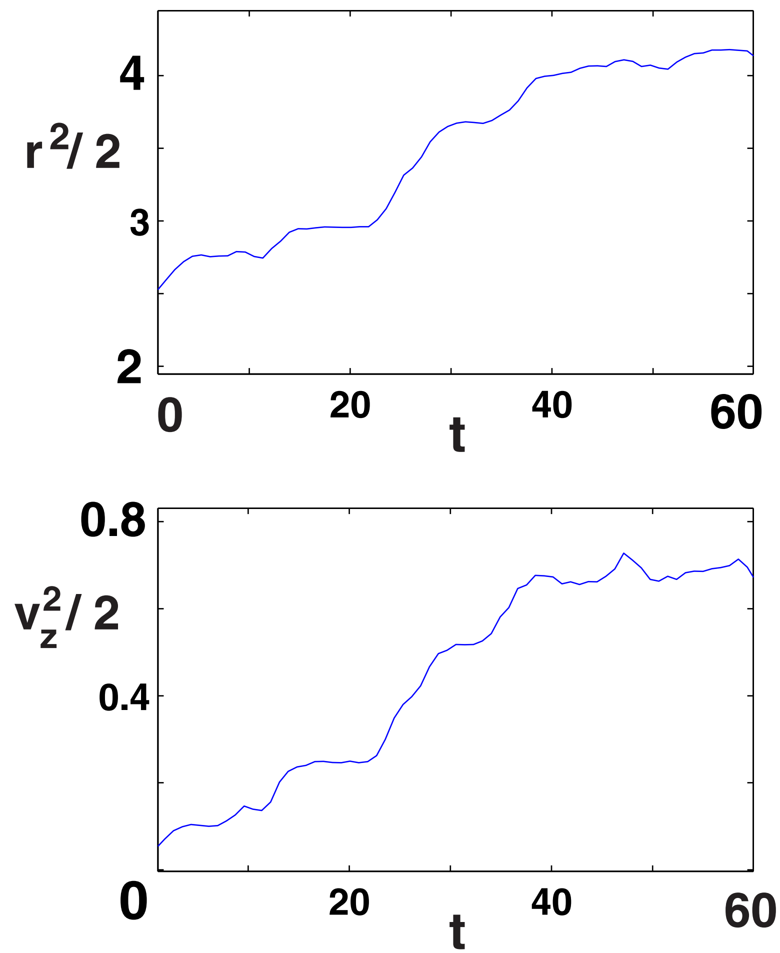
Short-Time Heating

$v_{z0} = 0.25, r_0 = 2.24$

$\epsilon = 0.4, \nu = 3.6$ 1 wave



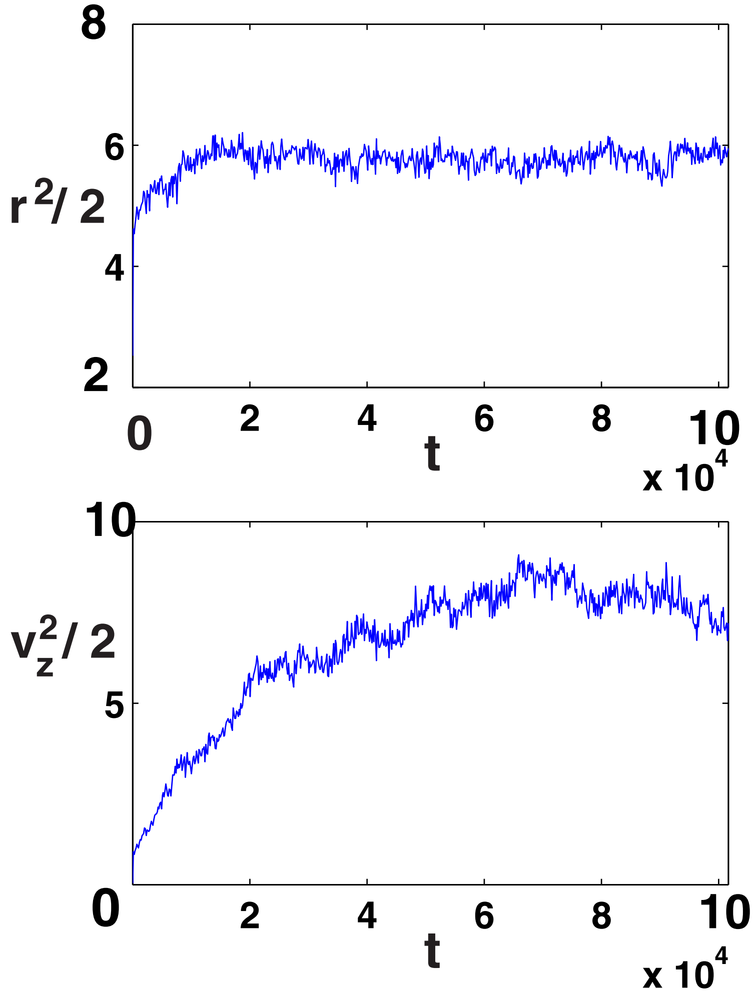
$\epsilon = 0.2, \nu = 3.6, 3.1$ 2 waves



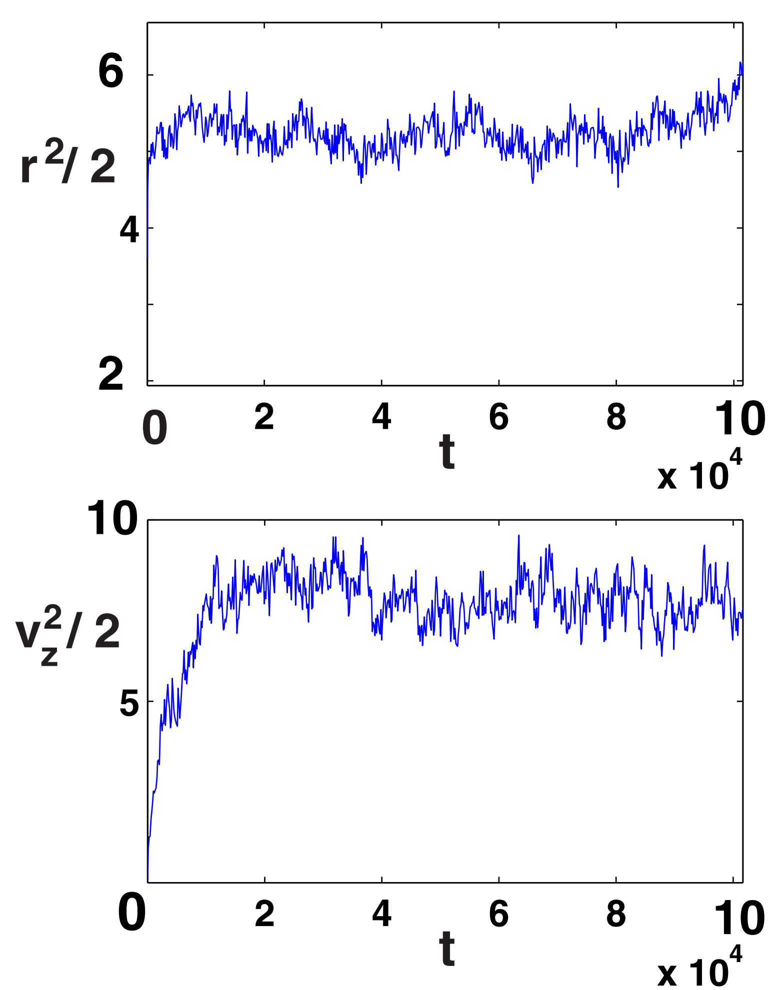
Long-Time Heating

$v_{z0} = 0.25, r_0 = 2.24$

$\epsilon = 0.4, \nu = 3.6$ 1 wave



$\epsilon = 0.2, \nu = 3.6, 3.1$ 2 waves



Future Work

- Coherent motion for ν_1, ν_2 both integers.
- Study of stochastic "diffusion" on short and long timescale.
- Investigation for realistic plasma and wave parameters.

References

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