COHERENT AND STOCHASTIC MOTION OF IONS IN TWO OBLIQUE ELECTROSTATIC WAVES

D. J. Strozzi, A. K. Ram, A. Bers

Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139. U.S.A.

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ABSTRACT

The stochastic motion of ions in the presence of a background magnetic field and electrostatic waves is of interest in both laboratory and space plasmas. Ion heating can be achieved by a single perpendicular or oblique wave driving particles into chaotic dynamics [C. Karney, *Phys. Fluids* 21,9 (1978), G. Smith, A. Kaufman, *Phys. Fluids* 21,12 (1978)].

A spectrum of multiple waves allows for phenomena not possible with a single wave. For instance, ions can be coherently accelerated from low perpendicular energies to the stochastic region by two waves whose frequencies ω_1, ω_2 differ by an integer multiple of the cyclotron frequency [D. Bénisti, A. K. Ram, A. Bers, *Phys. Plasmas*, 5,9 (1998)]:

$$\omega_1 - \omega_2 = N\omega_{ci}$$

It has recently been shown that two perpendicular waves may explain the high-energy tail of H^+ and O^+ distributions in the upper ionosphere [A. K. Ram, A. Bers, D. Bénisti, *J. Geophys. Res.*, 103,A5 (1998)].

We show that waves with finite parallel wavenumbers k_{1z}, k_{2z} can also produce coherent acceleration. This occurs provided the parallel wavenumbers are sufficiently close to each other, regardless of how large they are. The resonance condition applies to the Doppler-shifted wave frequencies:

$$(\omega_1 - k_{1z}v_z) - (\omega_2 - k_{2z}v_z) = N\omega_{ci}$$

A nonzero $k_{1z} - k_{2z}$ leads to coherent motion in v_z as well as perpendicular energy. A change in v_z leads to a breakage of the resonance condition, and leads to a severe limitation of the coherent motion. This is similar to what happens for frequencies that do not differ by exactly an integer multiple of ω_{ci}.

OUTLINE

- One Perpendicular Wave: Stochastic region for gyroradius $k_\perp \rho \gtrsim \omega/\omega_{ci}$

- $\omega_1 - \omega_2 = N\omega_{ci}$: Resonant Hamiltonian which describes coherent motion

- Two Perpendicular Waves
  - Coherent acceleration of low-energy ions to stochastic region
  - Coherent range in ρ scales linearly with wave frequency
  - Period of coherent oscillation scales like ω_1^4
  - Departure from resonance: $\omega_1 - \omega_2 = N\omega_{ci} + \Delta\omega$, "bandwidth" $\Delta\omega$ where coherent motion persists scales like ω_1^{-4}

- Two Oblique Waves
  - Stochastic Motion: Lower bound in ρ similar to one-wave case, but upper bound lower; motion in ρ and v_z stochastic
  - $k_{1z} = k_{2z}$: Coherent motion in ρ persists, similar to perpendicular waves
  - $k_{1z} \neq k_{2z}$: v_z evolves coherently
  - $k_{1z} \neq k_{2z}$: Small difference $k_{1z} - k_{2z}$ can severely limit coherent energization: Ion sees Doppler-shifted wave frequencies, which depart from resonance as v_z evolves coherently

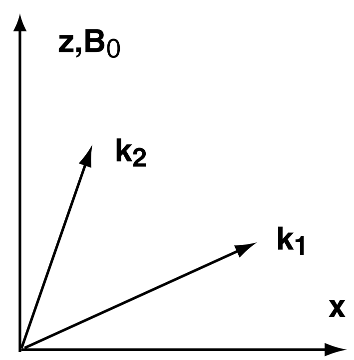
Equations of Motion

- Ion moving in two electrostatic waves and uniform $\vec{B}$:

$$M\frac{d^2\vec{x}}{dt^2} = q\sum_{i=1}^{2}\Phi_i\vec{k}_i\sin(\vec{k}_i\cdot\vec{x}-\omega_i t)+q\vec{v}\times\vec{B}$$

- Normalizations: time to $\omega_{ci}\equiv qB_0/M$, distances to k_{1x}.

$$\nu_i\equiv\frac{\omega_i}{\omega_{ci}}\qquad \epsilon_i\equiv\left(\frac{\omega_{Bi}}{\omega_{ci}}\right)^2\qquad \omega_{Bi}\equiv\sqrt{\frac{qk_{1x}^2\Phi_i}{M}}$$



- Hamiltonian formulation:

$$h(\vec{p},\vec{x},t)=\tfrac{1}{2}(\vec{p}-x\hat{y})^2+\sum_i\epsilon_i\cos(k_{ix}x+k_{iz}z-\nu_i t)$$

- Gyro-Variables:

$$\rho\equiv\sqrt{v_x^2+v_y^2}=\text{gyroradius}\qquad I\equiv\frac{\rho^2}{2}=\text{perp. energy}\qquad \phi\equiv\arctan\left(-\frac{v_y}{v_x}\right)=\text{gyrophase}$$

- Hamiltonian for gyro-variables:

$$H(I,v_z,\phi,z,t)=I+\tfrac{1}{2}v_z^2+\sum_i\epsilon_i\cos(k_{ix}\rho\sin\phi+k_{iz}z-\nu_i t)$$

One Perpendicular Wave: Stochastic Region in ρ

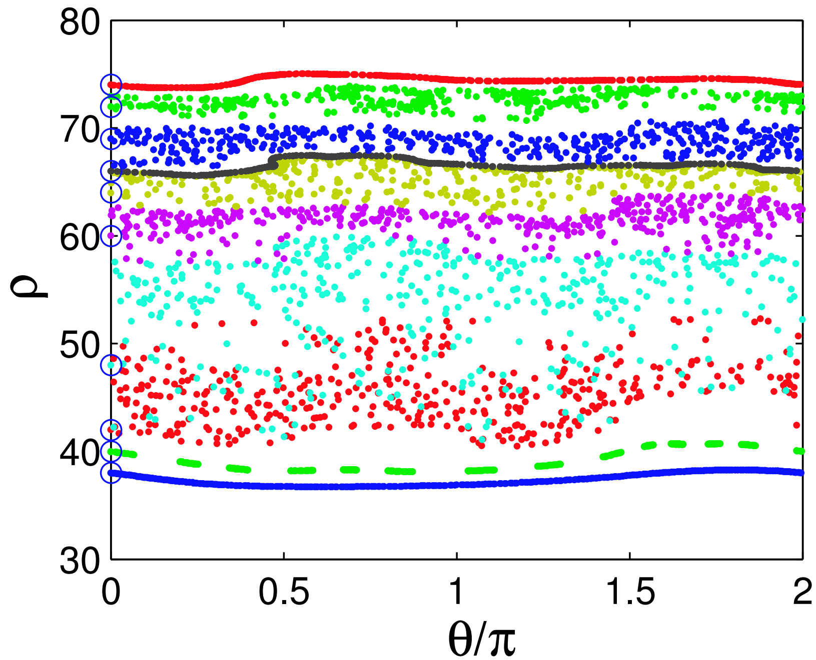
[C. Karney, *Phys. Fluids* 21,9 (1978), C. Karney, A. Bers, *Phys. Rev. Lett.* 39,9 (1977)]

$$H = I + \epsilon \cos(\rho \sin \phi - \nu t)$$

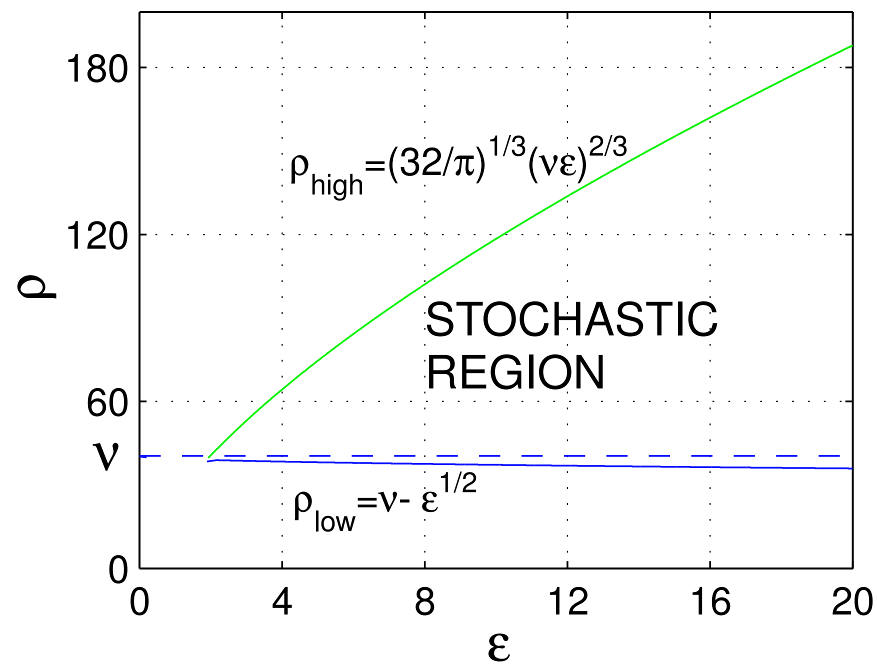
Surface of Section: $\phi = 2\pi n$

$\theta \equiv (\nu t) \mod (2\pi)$

$\nu = 40.37, \epsilon = 4$



Stochastic Region



- Ions below stochastic region $\rho \gtrsim \nu - \sqrt{\epsilon}$ not energized.

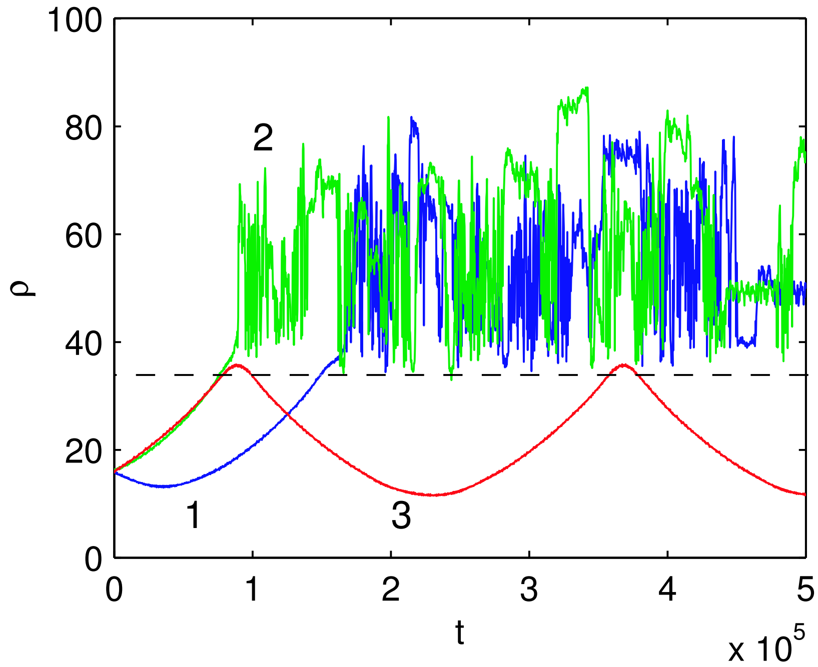
Coherent Motion when $\nu_1 - \nu_2$ an Integer

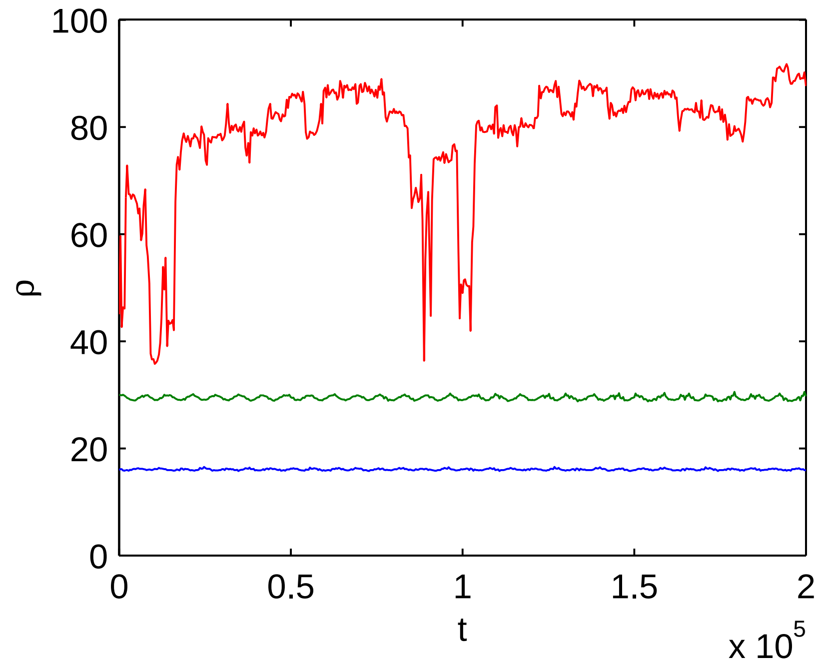
[D. Bénisti, A. K. Ram, A. Bers, *Phys. Plasmas* 5,9 (1998)]

Second-order resonance: $\nu_1 - \nu_2 \equiv N =$ integer

Initial condition: $v_{z0} = 0$

$\epsilon_1 = \epsilon_2 = 4 \qquad k_{1x} = k_{2x} = 1 \qquad k_{1z} = k_{2z} = 0$





$\nu_1 = 40.37 \qquad \nu_2 = 39.37$

$\nu_1 - \nu_2 =$ integer

$\nu_1 = 40.37 \qquad \nu_2 = 39.369$

$\nu_1 - \nu_2 \neq$ integer

Coherent Motion: Lie Perturbation Technique

[A. Lichtenberg, M. Lieberman, "Regular and Stochastic Motion." Springer-Verlag (1992), J.Cary, *Physics Reports* 79,2 (1981)]

$x = (p, q) =$ physical coordinates $\qquad \bar{x} = (\bar{p}, \bar{q}) =$ new coordinates

$\bar{H}(\bar{x})$ describes resonant motion in $H(x)$

Dependence on perturbation parameter ϵ, ensures $x \longrightarrow \bar{x}$ is canonical:

$$\frac{\partial \bar{x}}{\partial \epsilon} = [\bar{x}, w(\bar{x}, t)]_{\bar{x}} \qquad \bar{x}(\epsilon = 0) = x$$

$$[f, g]_x = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) = \text{ Poisson bracket} \qquad w = \text{ "Lie generating function"}$$

Coordinate transformation:

$$(Tf)(x) = f(\bar{x}) \qquad f(\bar{x}) = \bar{x} \longrightarrow Tx = \bar{x}$$

Equation for T:

$$\frac{\partial T}{\partial \epsilon} f(x) = -T[w(x, t), f(x)]_x$$

Transformed Hamiltonian:

$$\boxed{\bar{H} = T_\epsilon^{-1} H + T_\epsilon^{-1} \int_0^\epsilon d\epsilon' T_{\epsilon'} \frac{\partial w}{\partial t}}$$

Deprit Perturbation Series

[A. Deprit, *Cel. Mech.* 1 (1969)]

$$H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \qquad \bar{H} = H_0 + \epsilon \bar{H}_1 + \epsilon^2 \bar{H}_2$$

Expand Lie generating function:

$$w = w_1 + \epsilon w_2$$

Coordinate change T must be near-identity:

$$\bar{x} = Tx = x - \epsilon[w_1, x] + O(\epsilon^2)$$

To keep T near-identity, w_i must remain small.

Expand $\bar{H}$ equation:

$$\begin{aligned} D_0 w_1 &= \bar{H}_1 - H_1 \\ D_0 w_2 &= 2(\bar{H}_2 - H_2) - [w_1, \bar{H}_1 + H_1] \end{aligned}$$

$$D_0 f = \frac{\partial f}{\partial t} + [f, H_0] = \text{derivative along unperturbed orbit}$$

Choose $\bar{H}_i$ to remove secularities in w_i equation.

Two-Wave Perturbation Theory

$$H = H_0 + H_1 \qquad H_0 = I + \tfrac{1}{2}v_z^2 \qquad H_1 = \sum_I \epsilon_i \cos(k_{ix}\rho\sin\phi + k_{iz}z - \nu_i t)$$

Unperturbed ($\epsilon_i = 0$) orbits: $I = \text{const.}, \quad v_z = 0, \quad \phi = t, \quad z = z_0.$

$O(\epsilon)$:

$$(\partial_t + \partial_\phi + v_z\partial_z)w_1 = \bar{H}_1 - H_1 = \bar{H}_1 - \sum_{i,m} \epsilon_i J_m(k_{ix}\rho)\cos(m\phi + k_{iz}z - \nu_i t)$$

No resonant terms:

$$\boxed{\bar{H}_1 = 0} \qquad w_1 = -\sum_{i,m} \frac{\epsilon_i J_m(k_{ix}\rho)}{m - (\nu_i - k_{iz}v_z)} \sin(m\phi + k_{iz}z - \nu_i t)$$

$O(\epsilon^2)$:

$$(\partial_t + \partial_\phi + v_z\partial_z)w_1 = 2\bar{H}_2 - [w_1, H_1]$$

$[w_1, H_1]$ gives terms containing

$$\cos\left[(m-n)\phi - (\nu_1 - \nu_2)t + (k_{1z} - k_{2z})z\right]$$

$\nu_1 - \nu_2 \in \mathcal{Z}$: Resonate along unperturbed orbits $\longrightarrow$ secular growth in w_2.

$$\boxed{\bar{H}_2 = \tfrac{1}{2}\,(\text{resonant terms in } [w_1, H_1])}$$

Second-Order Hamiltonian for Coherent Motion

Coherent Hamiltonian: $\bar{H}(\bar{I}, \bar{v}_z, \bar{\psi}, \bar{z}) = \frac{1}{2}\bar{v}_z^2 + S_0(\bar{I}, \bar{v}_z) + S_-(\bar{I}, \bar{v}_z)\cos(N\bar{\psi} + \Delta k_z \bar{z})$

$\bar{\psi} = \bar{\phi} - t =$ angle in rotating gyro-frame

$N = \nu_1 - \nu_2 \qquad \Delta k_z = k_{1z} - k_{2z}$

S_0, S_- are second-order in wave amplitudes: $S_0 \sim \epsilon_1^2, \epsilon_2^2 \qquad S_- \sim \epsilon_1 \epsilon_2$

- Barred coordinates differ from physical coordinates by incoherent fluctuations, e.g.:

$$I = \bar{I} - \epsilon_i \sum_m \frac{m J_m(k_{ix}\bar{\rho})}{m - \nu_i} \cos(m\bar{\phi} + k_{iz}\bar{z} - \nu_i t) + O(\epsilon_i^2)$$

Coherent Motion in v_z

- $\bar{H}$ is a constant of the motion. Second constant of the motion:

$$\frac{d}{dt}\left(\bar{v}_z - \frac{\Delta k_z}{N}\bar{I}\right) = 0 \qquad \rightarrow \qquad \bar{v}_z = v_{z0} + \frac{\Delta k_z}{N}(\bar{I} - I_0)$$

Bounds of coherent motion

- $\bar{v}_z$ is a function of $\bar{I}$:

$$\bar{H} = \tfrac{1}{2}\bar{v}_z(\bar{I})^2 + S_0(\bar{I}) + S_-(\bar{I})\cos(N\bar{\psi} - \Delta k_z \bar{z})$$

$$\cos(N\bar{\psi} - \Delta k_z \bar{z}) = \frac{\bar{H} - \frac{1}{2}\bar{v}_z^2 - S_0}{S_-}$$

$$|\cos x| \leq 1 \qquad \rightarrow \qquad |\bar{H} - \tfrac{1}{2}\bar{v}_z^2 - S_0| > |S_-| \qquad \text{forbidden}$$

Potential barriers:

$$\boxed{H_\pm(\bar{I}) = \tfrac{1}{2}\bar{v}_z^2 + S_0 \pm |S_-| \qquad\qquad H_- \leq \bar{H} \leq H_+}$$

Turning points in $\bar{I}$:

$$\bar{H} = \tfrac{1}{2}\bar{v}_z^2 + S_0 \pm |S_-|$$

Occur when

$$N\bar{\psi} - \Delta k_z \bar{z} = m\pi$$

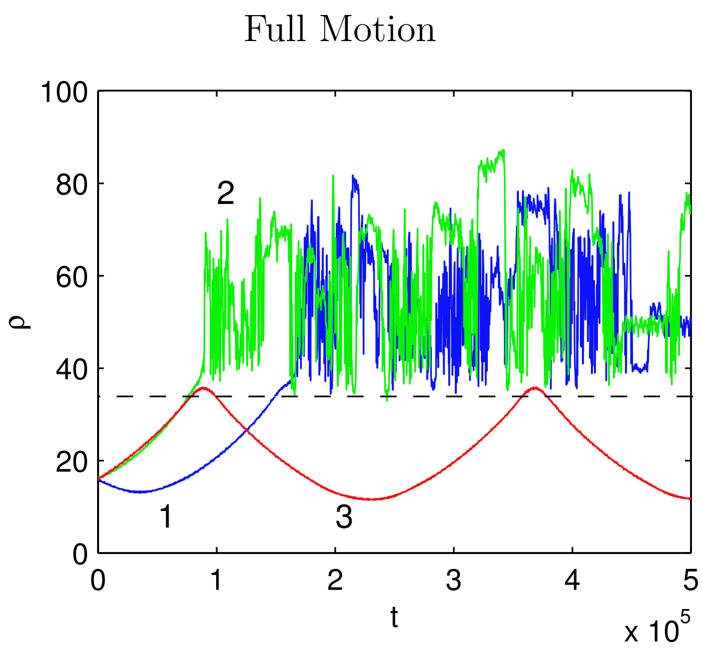
Expressions for the S's

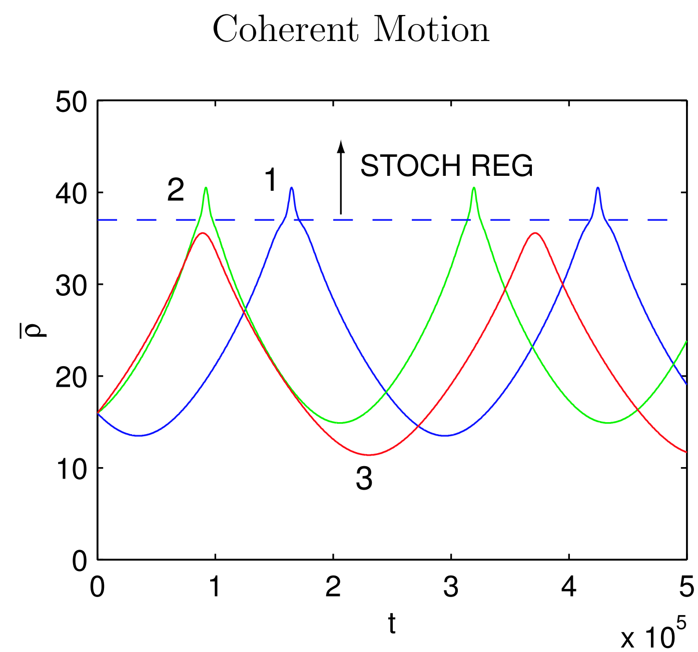
$$
\begin{aligned}
S_0 &= S_{0x} + S_{0z} \\
S_{0x} &= -\frac{1}{2\bar{\rho}} \sum_i k_{ix} \epsilon_i^2 \frac{m}{m-\mu_i} J_{m,i} J'_{m,i} \\
&= \frac{\pi}{8} \sum_i k_{ix}^2 \epsilon_i^2 \frac{J_{\mu_i+1,i} J_{-\mu_i-1,i} - J_{\mu_i-1,i} J_{-\mu_i+1,i}}{\sin \pi \mu_i} \\
S_{0z} &= \frac{1}{4} \sum_i k_{iz}^2 \epsilon_i^2 \frac{1}{(m-\mu_i)^2} J_{m,i}^2 \\
&= -\frac{\pi}{4} \sum_i k_{iz}^2 \epsilon_i^2 \frac{\partial}{\partial \mu_i} \frac{J_{\mu_i,i} J_{-\mu_i,i}}{\sin \pi \mu_i}
\end{aligned}
$$

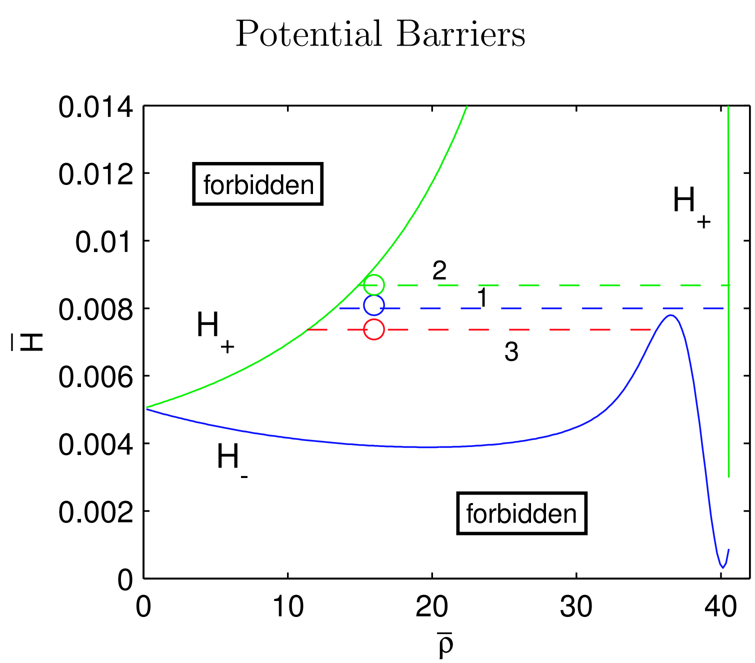
$$
\begin{aligned}
S_- &= S_{-x} + S_{-z} \\
S_{-x} &= -\frac{1}{4\bar{\rho}} \epsilon_1 \epsilon_2 \left(\frac{1}{m-\mu_1} + \frac{1}{m-\mu_3} \right) \times \\
&\quad \times \left(k_{1x}(m-N) J'_{m,1} J_{m-N,2} + k_{2x} m J_{m,1} J'_{m-N,2} \right) \\
S_{-z} &= \frac{1}{4} k_{1z} k_{2z} \epsilon_1 \epsilon_2 \left(\frac{1}{(m-\mu_1)^2} + \frac{1}{(m-\mu_3)^2} \right) J_{m,1} J_{m-N,2}
\end{aligned}
$$

$$
\mu_i = \nu_i - k_{iz}\bar{v}_z \text{ for } i = 1,2 \qquad \mu_3 = \nu_1 - k_{2z}\bar{v}_z \qquad J_{m,i} = J_m(k_{ix}\bar{\rho})
$$

Two Perpendicular Waves: Coherent ρ Motion





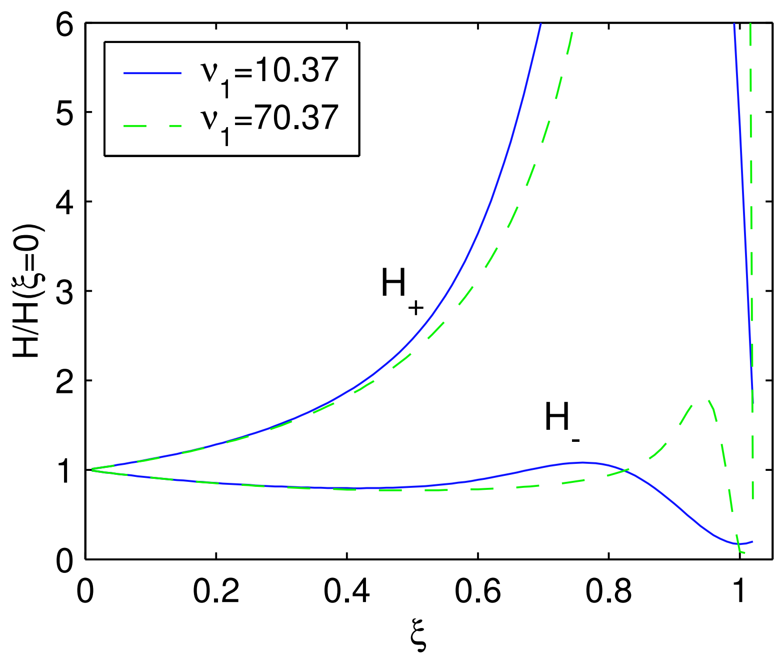


Two Perpendicular Waves: Scaling with wave frequencies

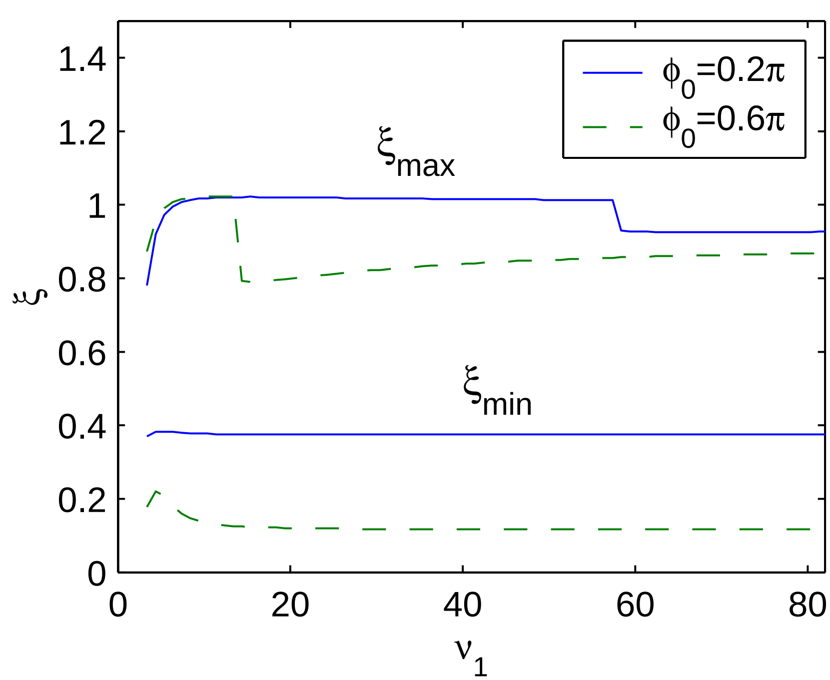
$$\xi \equiv \frac{\bar{\rho}}{\nu_1}$$

- Range of motion in ξ does not change much with ν_1.

Rescaled $\tilde{H}$ vs. ξ for $\nu_1 - \nu_2 = 1$



Coherent range of motion in ξ for $\xi_0 = 0.4$



Period of Coherent Oscillation for Perpendicular Waves

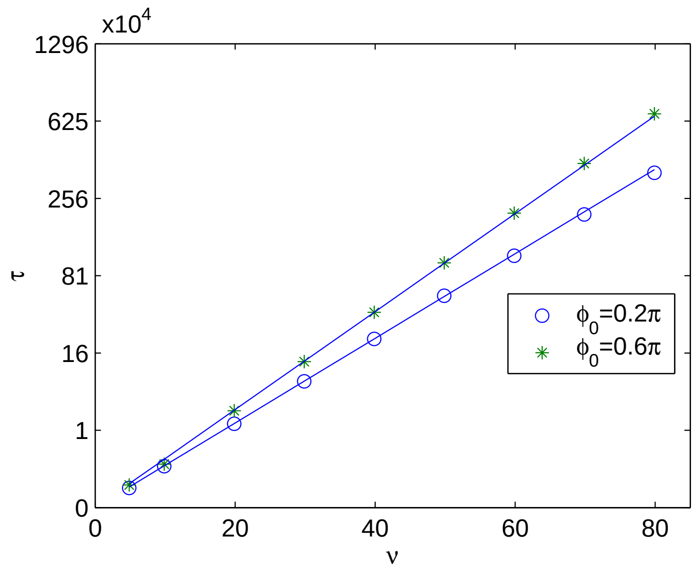
- coherent motion is oscillatory in $\bar{I}$ with a period $\gg$ cyclotron period.

For $\nu_1 \approx \nu_2$ and $\epsilon_1 = \epsilon_2$,

Period scaling: $$\tau \equiv \frac{2\pi}{N\langle d\bar{\psi}/dt\rangle} \sim \frac{\nu_1^4}{N\epsilon_1^2}$$

- Increasing wave frequency vastly increases period of coherent motion.

Period for Orbits of $\tilde{H}$ vs. ν_1^4 scaling
lines match τ at $\nu_1 = 40.37$



$\nu_1 - \nu_2$ not an Integer: "Bandwidth" for Coherent Motion

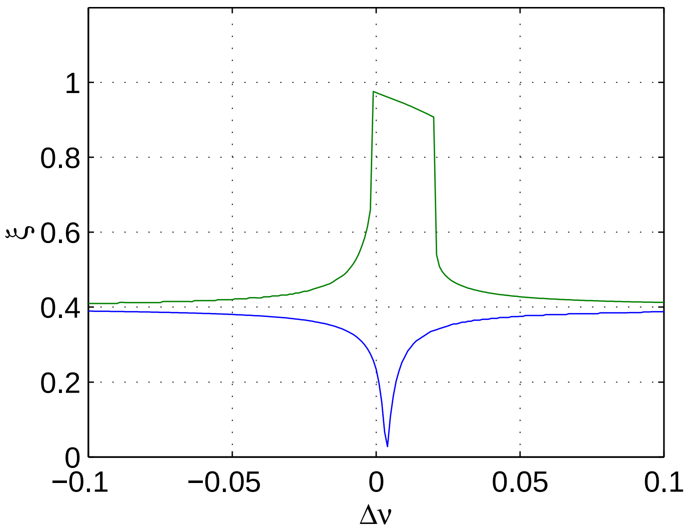
$\nu_1 - \nu_2 = N + \Delta\nu$: resonant terms for $\Delta\nu = 0$ are near-resonant

$$\boxed{\bar{H} = -\Delta\nu \bar{I} + S_0 + S_- \cos N\bar{\psi}}$$

$\Delta\nu_*$: Critical $\Delta\nu$ when $-\Delta\nu\bar{I}$ dominates over S_0 term and starts limiting motion

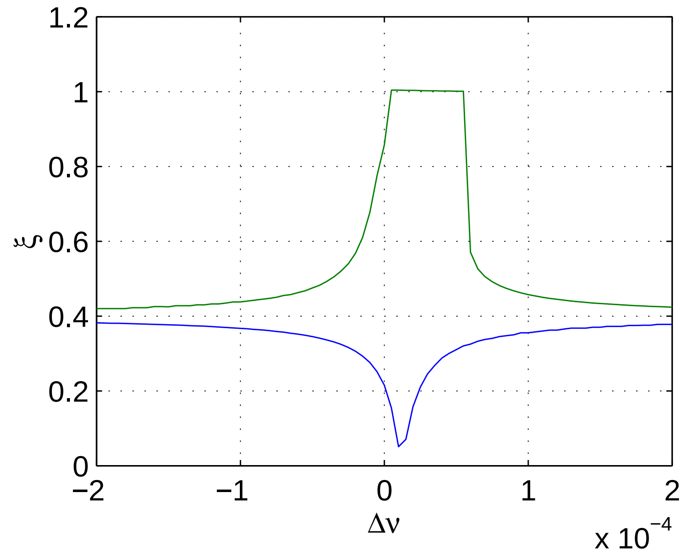
$$\boxed{\Delta\nu_* \sim \frac{\epsilon_1^2 k_{1x}^2}{\nu_1^4}}$$

ξ Range for $\boxed{\nu_1 = 10.37}$, $N = 1$ vs. $\Delta\nu$ for $\xi_0 = 0.4, \phi_0 = \pi/2$



Plateau stops at $\Delta\nu_* = 0.02$

ξ Range for $\boxed{\nu_1 = 40.37}$, $N = 1$ vs. $\Delta\nu$ for $\xi_0 = 0.4, \phi_0 = \pi/2$



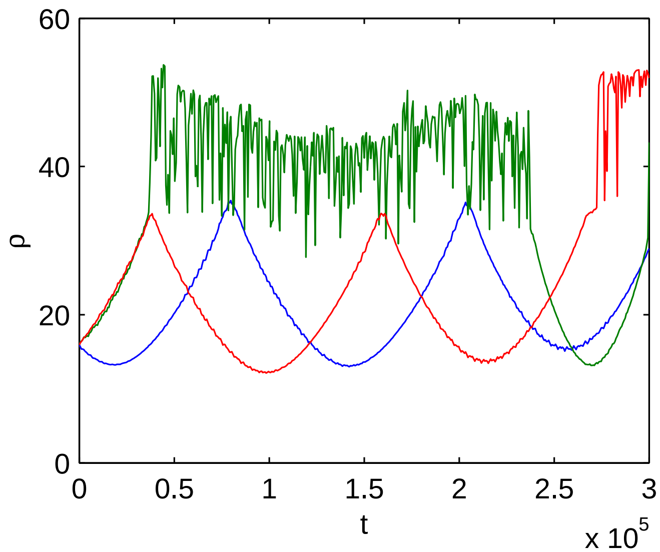
Plateau stops at $\Delta\nu_* = 6 \cdot 10^{-5}$

$0.02/6 \cdot 10^{-5} = 333 \qquad (40.37/10.37)^4 = 230$

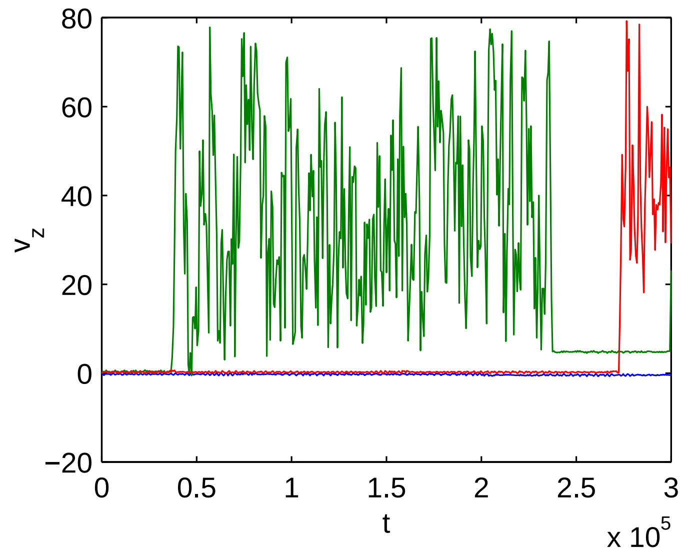
Oblique Waves with $k_{1z} = k_{2z}$: Coherent Motion in ρ

$\bar{v}_z = v_{z0} + \frac{\Delta k_z}{N}(\bar{I} - I_0)$ $\Delta k_z = 0$: no coherent motion in v_z

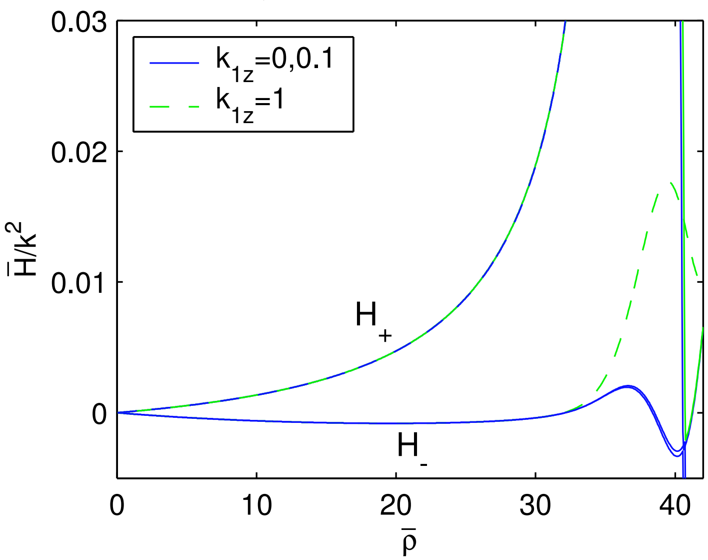
ρ vs. t for 45° waves



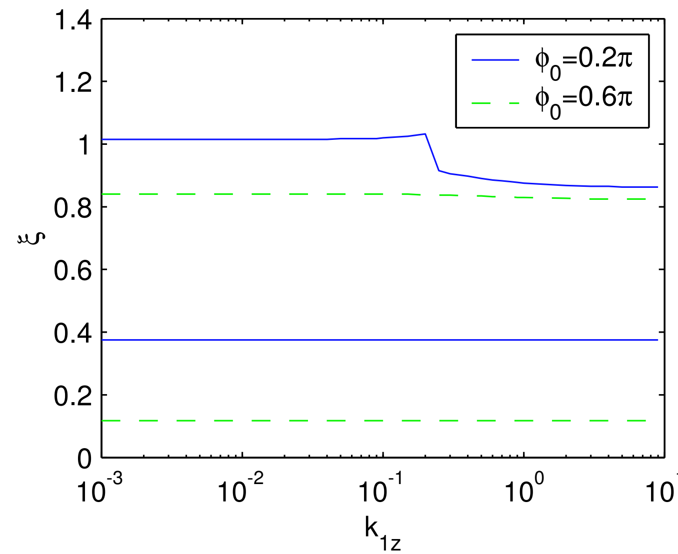
v_z vs. t for 45° waves



$H_{\pm}$ vs. $\bar{\rho}$ for $k_{1z} = 0, 0.1, 1$



Range of ξ Motion vs. k_{1z} for $\xi_0 = 0.4$

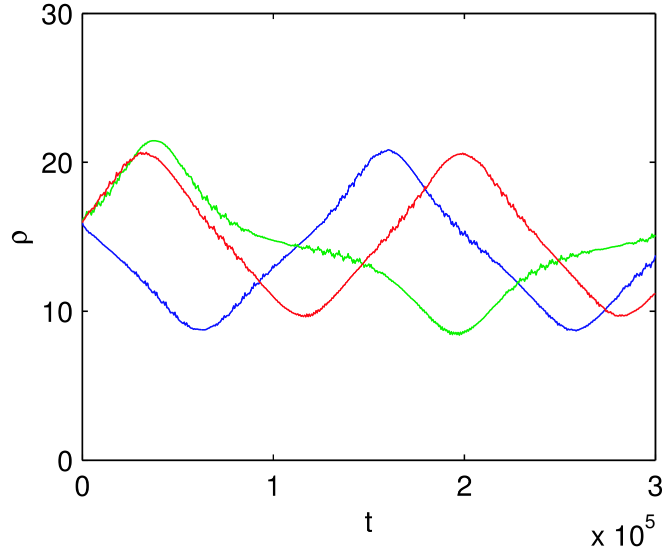


Oblique Waves with $k_{1z} \neq k_{2z}$: Coherent Motion Limited

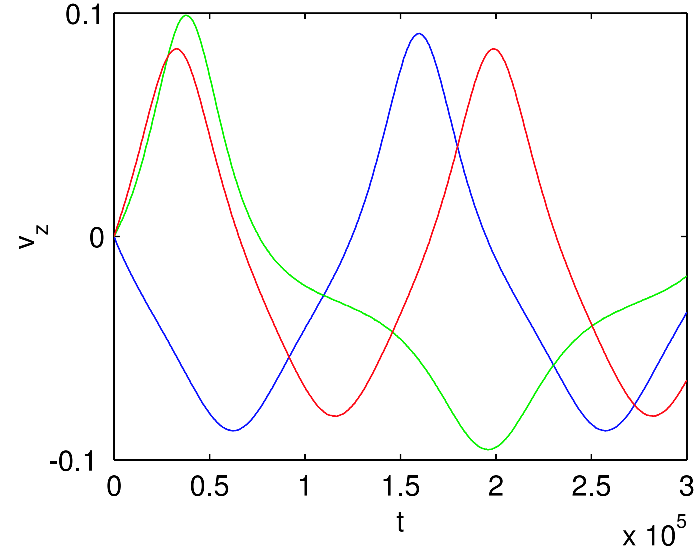
$\bar{v}_z = v_{z0} + \frac{\Delta k_z}{N}(\bar{I} - I_0)$

$\Delta k_z \neq 0$: small coherent motion in v_z

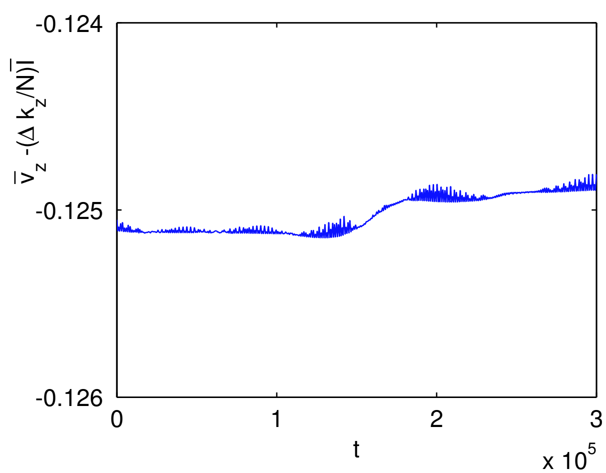
ρ vs. t for $k_{1z} = 0.001, k_{2z} = 0$



v_z vs. t for $k_{1z} = 0.001, k_{2z} = 0$



Second-Order Constant $\bar{v}_z - \frac{\Delta k_z}{N}\bar{I}$



Variation $< 0.3\%$

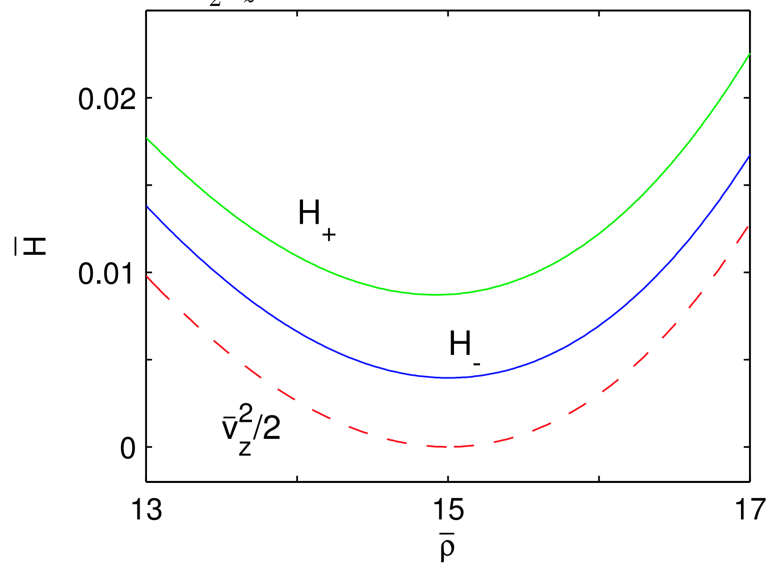
$k_{1z} \neq k_{2z}$: Potential Barriers Pulled Closer

$$\bar{H} = \frac{1}{2}\underbrace{\frac{\Delta k_z^2}{N^2}(\bar{I}-\bar{I}_0)^2}_{\bar{v}_z^2} + S_0 + S_- \cos(N\bar{\psi} - \Delta k_z \bar{z})$$

Δk_{z*} : Critical Δk_z when motion limited

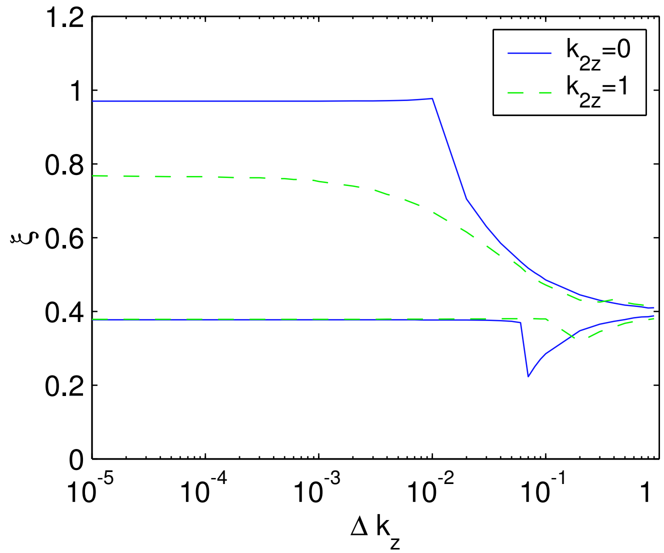
$$\boxed{\Delta k_{z*} \sim \frac{N\epsilon_1 k_{1x}}{\nu_1^3}}$$

$H_\pm, \frac{1}{2}\bar{v}_z^2$ for $k_{1z} = 0.005, k_{2z} = 0$

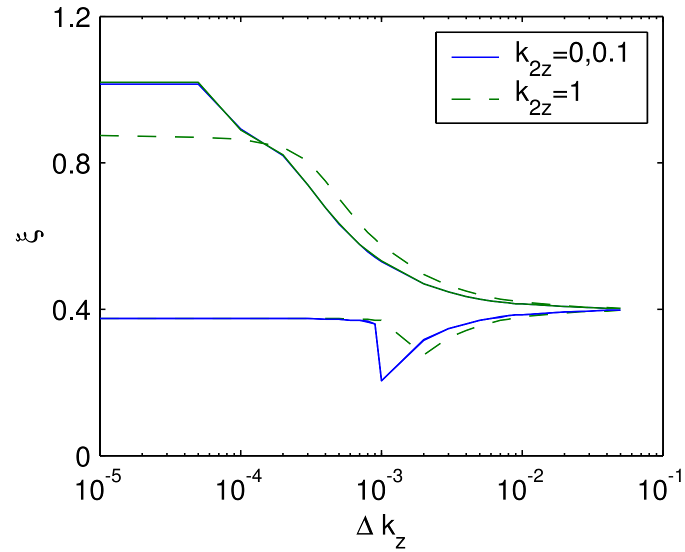


As Δk_z increases, a smaller change in $\bar{I}$ moves ions between the potential barriers H_- and H_+

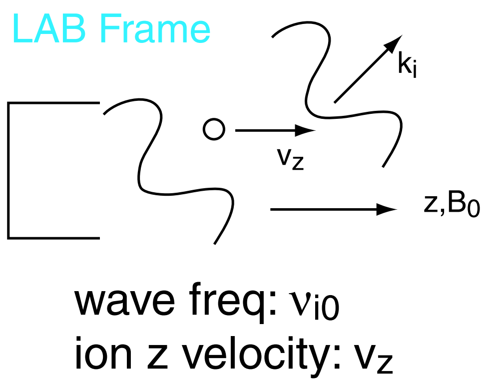
ξ range Δk_z for $\boxed{\nu_1 = 10.37}$, $N = 1, \xi_0 = 0.4$

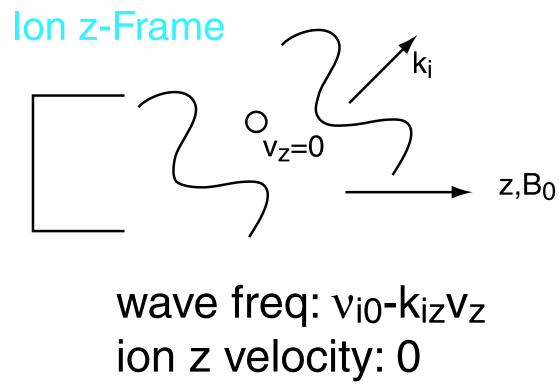


ξ range Δk_z for $\boxed{\nu_1 = 40.37}$, $N = 1, \xi_0 = 0.4$



$k_{1z} \neq k_{2z}$: Departure from $\nu_1 - \nu_2 =$ integer





Ion sees Doppler-shifted frequencies: $\nu_i = \nu_{i0} - k_{iz} v_z$

Resonance condition: $\boxed{\nu_1 - \nu_2 = \nu_{10} - \nu_{20} - \Delta k_z v_z = N}$

- $\Delta k_z \neq 0$: v_z changes coherently, resonance condition cannot stay satisfied.

Heating a Distribution in v_z

Lab frame: $v_z(t=0) = v_{z0}$

Resonance condition: $\nu_{10} - \nu_{20} - \Delta k_z v_{z0} = N$

- $\Delta k_z = 0$: Resonance for $\nu_{10} - \nu_{20} = N$. All ions resonate. Ions with larger $k_{iz} v_{z0}$ see smaller ν_i.
- $\Delta k_z \neq 0$: Only certain v_{z0} resonate: $\boxed{v_{z0N} = \frac{\nu_{10} - \nu_{20} - N}{\Delta k_z} \qquad N = 1, 2, 3, ...}$

CONCLUSIONS

- Coherent Energization to stochastic region possible for oblique waves provided $(k_{1z} - k_{2z})$ is small
- $k_{1z} \neq k_{2z}$ leads to coherent motion in v_z, which Doppler shifts waves away from resonance
- Lower wave frequencies ω_1, ω_2 are "more favorable:"
  - Departure of $(\omega_1 - \omega_2)/\omega_{ci}$ from an integer, and k_{1z} from k_{2z}, that still permit coherent energization scale like ω_1^{-4} and ω_1^{-3}, respectively
  - Period of coherent oscillation scales like ω_1^4

Questions? Comments? Reprints?