Collisionless Hall-MHD Modeling Near a Magnetic Null

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Collisionless Magnetic Reconnection

Magnetic reconnection refers to changes in the structure of magnetic fields, brought about by processes outside of ideal MHD. A simple example is a change in magnetic field topology due to resistivity: a current sheet forms, and field lines diffuse within it (eg, the Sweet-Parker model).

Many astrophysical and laboratory instances of reconnection occur in basically collisionless regimes. We need a mechanism to explain reconnection without resistivity or other collisional effects, and ultimately predict the reconnection rate.

We use a fluid model with a collisionless Ohm's law that includes an electron pressure and Hall term. We show that this system's equilibrium can be described by two coupled Poisson-like equations. We solve this system numerically for an imposed poloidal cusp field. This system has nontrivial plasma properties, including the possibility to break the frozen-in law.

Our model applies in parameter regimes where d_i~system size L and d_e<<L, such as those of the Versatile Toroidal Facility (VTF) experiment at MIT lead by A. Fasoli.

Governing Equations

Continuity: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

Momentum: $\rho \dfrac{d\vec{v}}{dt} = \vec{j} \times \vec{B} - \nabla p$

Ohm's Law: $\vec{E} + \vec{v} \times \vec{B} = \dfrac{1}{enm_e} \vec{j} \times \vec{B} - \dfrac{1}{en} \nabla p_e + \dfrac{m_e}{e} \dfrac{D\vec{v}_e}{Dt}$

State: $T_i = 0$, cold ions

$\vec{B} \cdot \nabla T_e = 0$, e^- isothermal along field lines

$\rightarrow$ total pressure $p = p_e + p_i \approx p_e = nT_e$

In equilibrium, $\partial / \partial t = 0$

Nondimensionalize equations using system size L, typical magnetic field B_0, density n_0, and Alfvén speed.

Continuity: $\nabla \cdot (\rho \vec{v}) = 0$

Momentum: $\vec{v} \cdot \nabla \vec{v} = \vec{j} \times \vec{B} - \beta_0 \nabla p$

Ohm's: $\vec{E} + \vec{v} \times \vec{B} = \dfrac{d_i}{L} \dfrac{1}{n} (\vec{j} \times \vec{B} - \nabla p) + \left(\dfrac{d_e}{L} \right)^2 *$ advective $\dfrac{D\vec{v}_e}{Dt}$ terms

$d_j =$ inertial skin depth $= \dfrac{c}{\omega_{pj}}, \quad \omega_{pj} = \sqrt{\dfrac{n_j q_j^2}{\varepsilon_0 m_j}}$

2-D Equilibrium, $\partial/\partial z=0$

$$\boxed{\vec{B} = \hat{z}\times\nabla\psi + B_z\hat{z}} \quad \rightarrow \quad \vec{j} = \nabla B_z \times \hat{z} + \nabla^2\psi\hat{z}$$

Let $\boxed{\vec{v} = \vec{u} + v_z\hat{z}}$

Integrate Continuity : $\nabla\cdot(\rho\vec{u}) = 0 \rightarrow$ $\boxed{\rho\vec{u} = \hat{z}\times\nabla\varphi}$

Temperature a flux function : $\vec{B}\cdot\nabla T = 0 \rightarrow$ $\boxed{T = T[\psi]}$

We drop terms of order $(d_e/L)^2$, as they are small in many regimes of interest (eg, VTF). Since $d_i \sim L$ in VTF, we retain these terms $\rightarrow$ non-ideal Ohm's Law.

Take z component and curl of Ohm's Law to eliminate electric field, which is $-\nabla\phi$ in equilibrium.

Reduction to Scalar PDEs

One can construct 4 equations of the form

$\nabla(\psi \text{ or } \varphi)\times\nabla f = 0 \rightarrow f = f[\psi \text{ or } \varphi]$. This gives:

$$d_i B_z = F[\psi] - \varphi \qquad\qquad d_i v_z = G[\varphi] + \psi$$

$$v_z = \frac{F'[\psi]}{\rho} B_z + U'[\psi] + \frac{d_i}{\rho}(j_z - \rho \log[\rho T'[\psi]])$$

$$\tfrac{1}{2} v^2 - \frac{U[\psi]}{d_i} + T[\psi](1 + \log\rho) = W[\varphi] \qquad \text{Bernoulli-like}$$

System reduces to 2 PDEs:

$$\nabla^2\psi = \rho T'[\psi]\log\rho + d_i^{-1}(-F'[\psi]B_z + \rho v_z - \rho U'[\psi])$$

$$\nabla\cdot(\frac{1}{\rho}\nabla\varphi) = -\frac{B_z}{d_i} - \frac{v_z G'[\varphi]}{d_i \rho} - \frac{W'[\varphi]}{\rho}$$

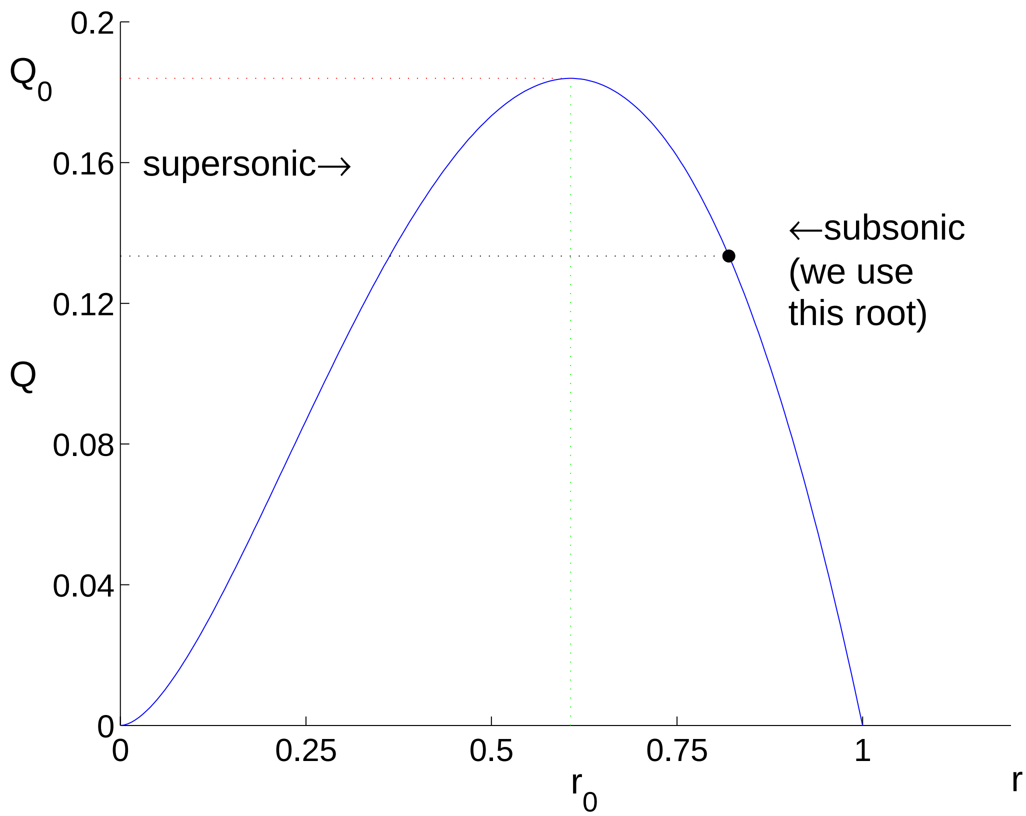
$+$ algebraic Bernoulli eqn. for ρ

Solving for ρ

$$\log\rho + a + \frac{b}{\rho^2} = 0, \qquad a = 1 + \frac{1}{T}\left(\tfrac{1}{2}v_z{}^2 - W[\varphi]\right), \quad \mathrm{b} = \frac{|\nabla\varphi|^2}{2T} \geq 0$$

$$\text{Let } r = \rho e^{a}, \quad Q = be^{2a} \quad \rightarrow$$

$$\boxed{-r^2 \log r = Q}$$



$$\text{At } r = r_0 = e^{-1/2}, Q = Q_0 = 1/2e = \text{maximum } Q$$

$$\text{Existence of solutions}: Q < Q_0$$

$$r \approx r_0 + \sqrt{Q_0 - Q} + A(Q_0 - Q)^{\alpha}$$

$$\text{poloidal Mach number } M^2 = \frac{u^2}{c_s{}^2} = \frac{|\nabla\varphi|^2}{\rho^2 T}$$

Free Functions

$$T[\psi] = \frac{\beta_0}{1+\Sigma\psi^2} + T_0, \qquad \text{we choose } \beta_0 = 0.01$$

$$F[\psi] = B_{z0} + F_1[\psi], \qquad F_1[\psi] = f_1 \tanh[\psi/\sigma]$$

$$G[\varphi] = -\frac{1}{\sigma}\operatorname{atanh}\left[\frac{\varphi}{F_1}\right] + G_0$$

$$U[\psi] = 0$$

$$W[\varphi] = T[F_1^{-1}[\varphi]]\{1 + K[F_1^{-1}[\varphi]]\} + G_0{}^2/2,$$

$$K[x] = \log\left[AT^{-1/2}[x]\right]$$

These choices ensure solutions of ρ equation exist on the boundary, where Q is largest.

Boundary Conditions

We want to simulate an externally imposed cusp field → $\boxed{\psi\big|_{bnd} = 2xy}$

Also choose $\varphi\big|_{bnd} = F_1[2xy]$. These choices give

$$B_z\big|_{bnd} = B_{z0}, \qquad v_z\big|_{bnd} = F_1[2xy]$$

$u_x\,|_{x=const} \propto \operatorname{sech}^2[2\sigma y] \rightarrow$ imposed normal flow drops off rapidly away from separatrix

Numerical Solution

We have two coupled Poisson-like PDEs for φ and ψ. We specify boundary conditions on a rectangle, and this determines the solution inside. We use finite differences, and treat the problem as a coupled, nonlinear system for φ and ψ at the discrete grid points.

Newton's method: quadratic convergence

Solve $f(u) = 0$, u = ψ or φ. Taylor expand about a guess, u_0, near solution u_* :

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + O((u - u_0)^2).$$

Since $f(u_*) = 0$,

$$f(u_*) = 0 \approx f(u_0) + f'(u_0)(u_* - u_0)$$

$$\boxed{u_* \approx u_0 - f'(u_0)^{-1} f(u_0)}$$

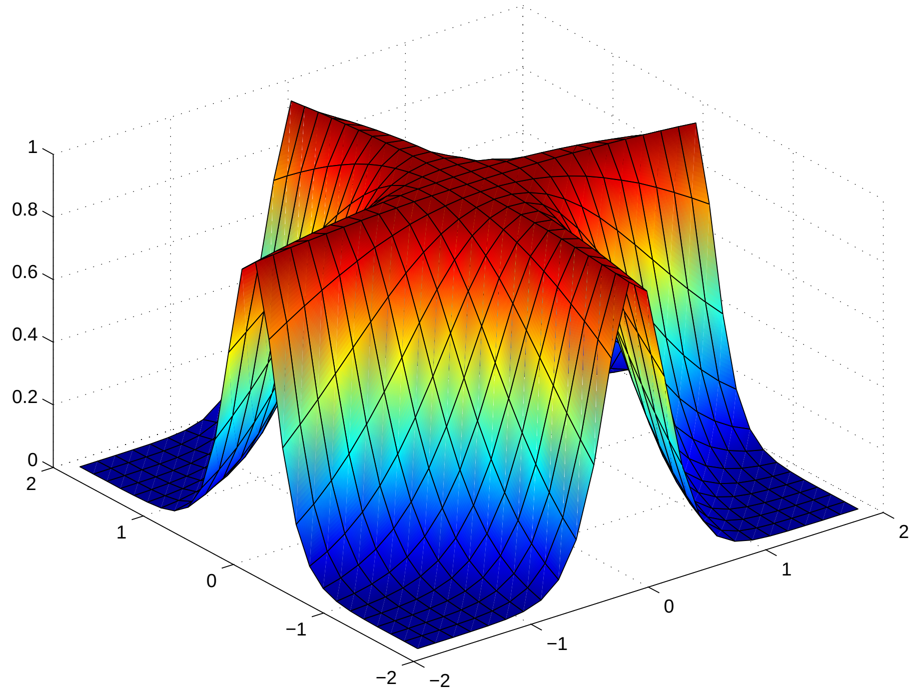
- Guess an initial u
- Compute the Jacobian $f'(u)^{-1}$
- Calculate new u
- iterate until average value of f(u) is ~ 0, given roundoff error.

Implemented in MATLAB. Works blindingly fast: 3 to 5 iterations for convergence.

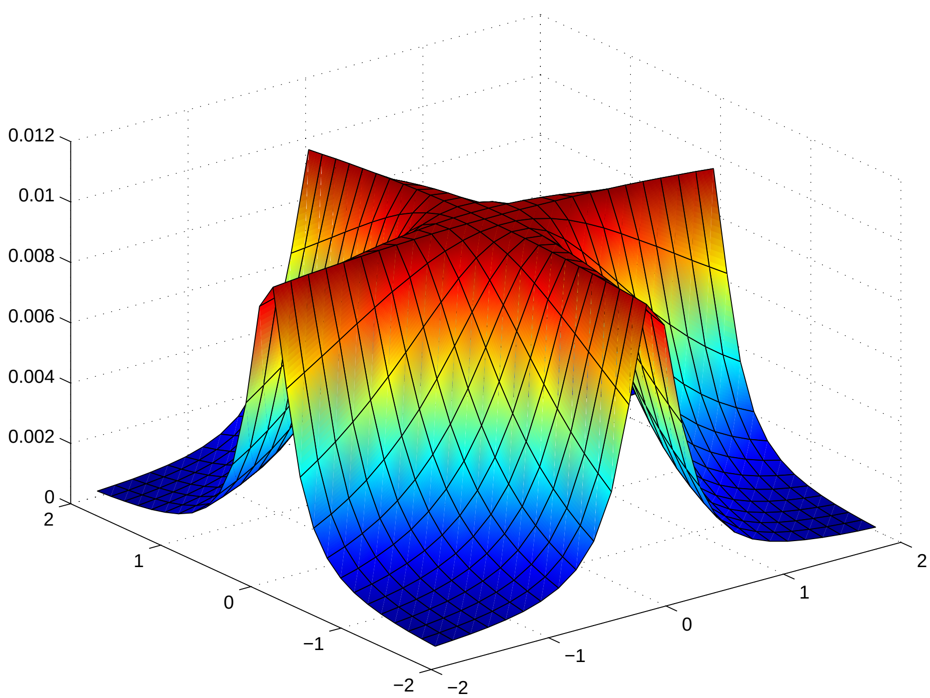
We set d_i=L and use a 2x2 domain in units of L. Discretized to 100x100 grid points.

B_{z0}=0 Equilibrium

Density: confined, almost flux function

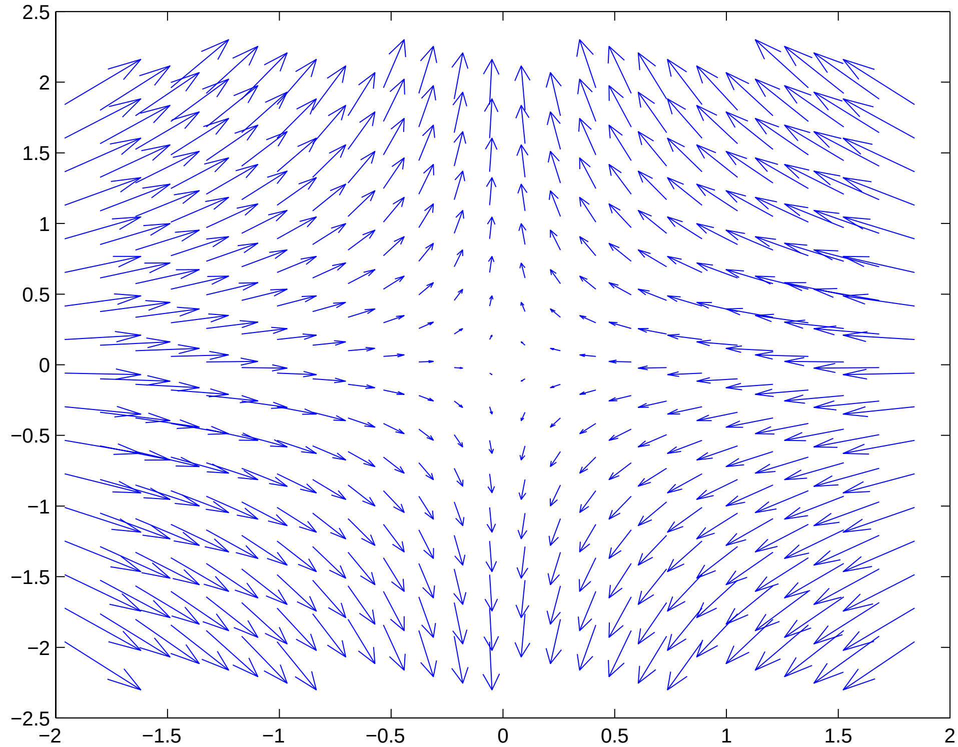


Temperature T[Ψ]

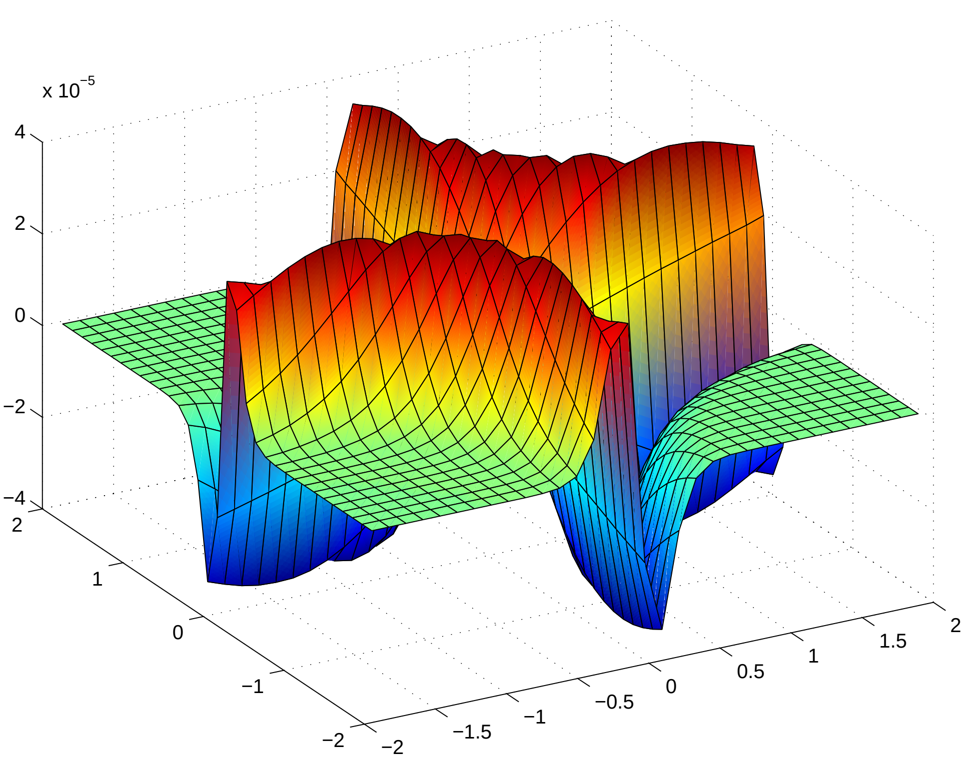


B_{z0}=0

Poloidal B: close to cusp

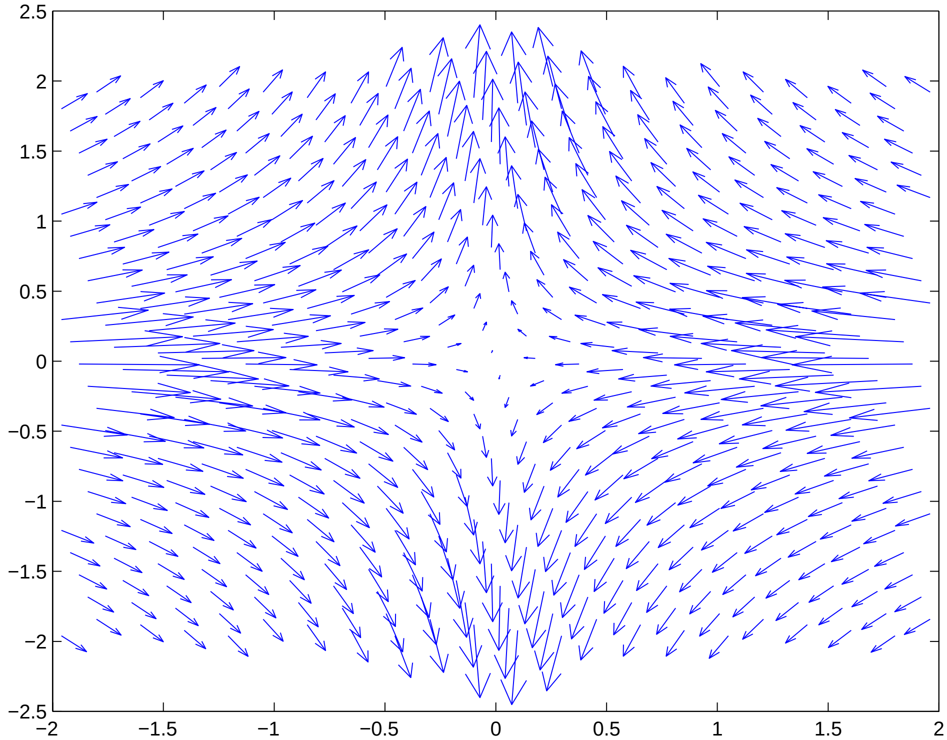


B_z

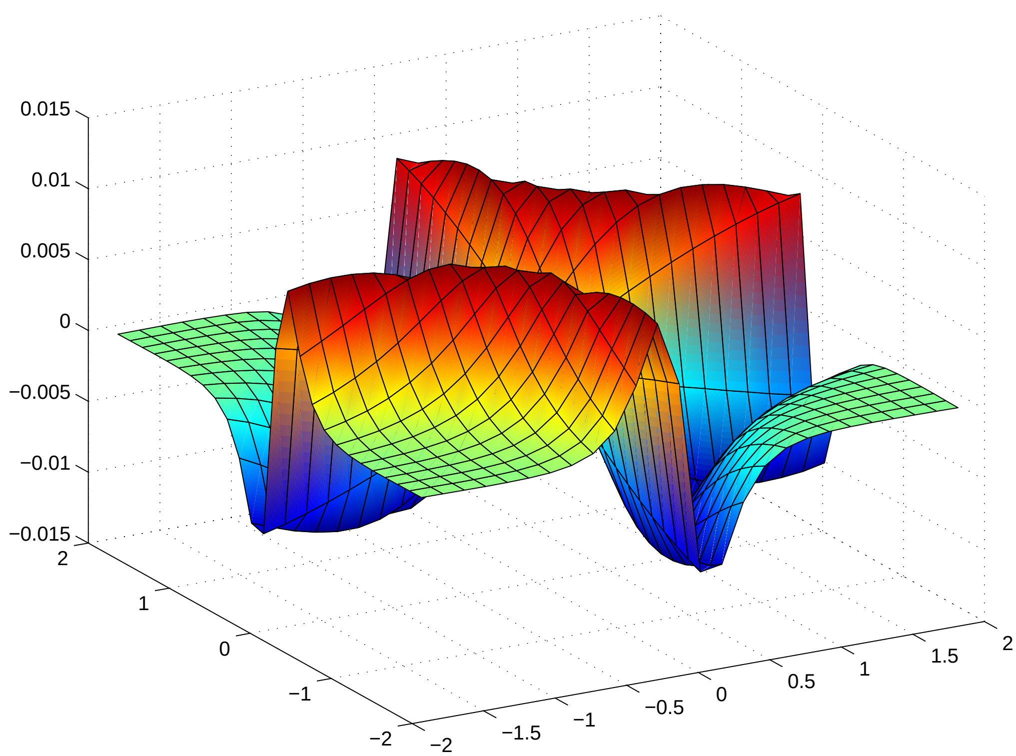


$B_{z0}=0$

Poloidal u (magnitude~ $0.01v_A$)

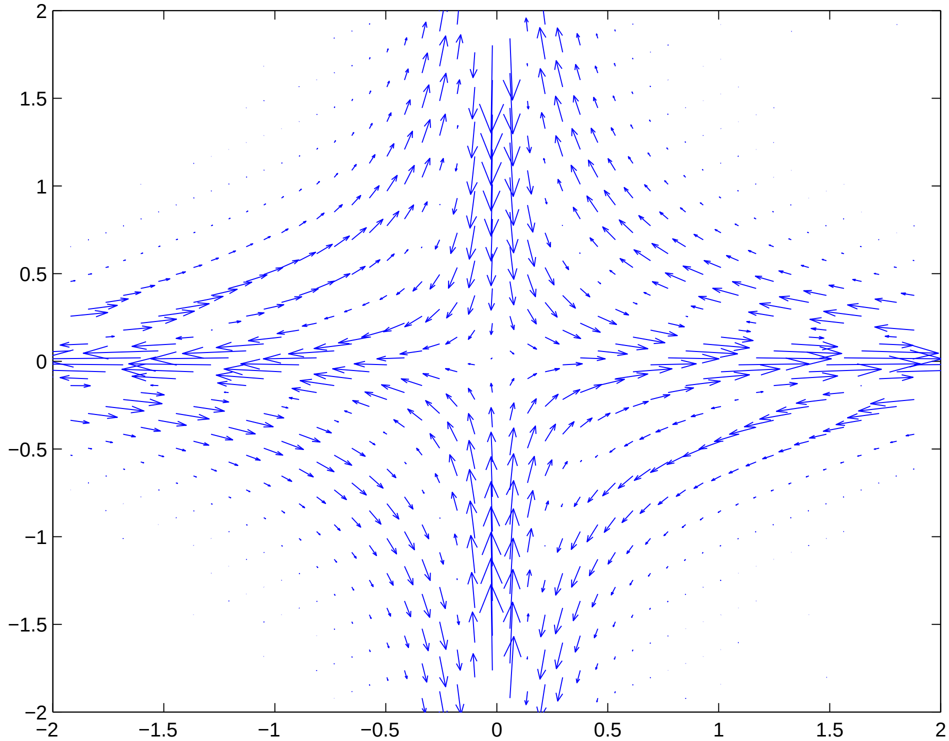


v_z

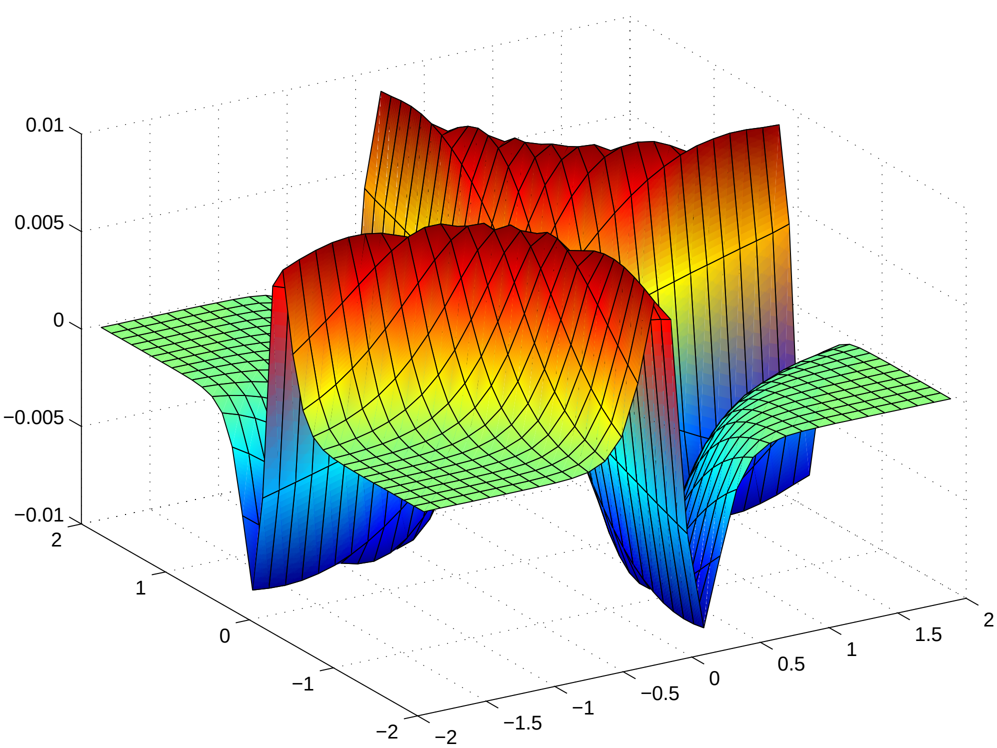


B_{z0}=0

Poloidal j: magnitude~10^{-4}



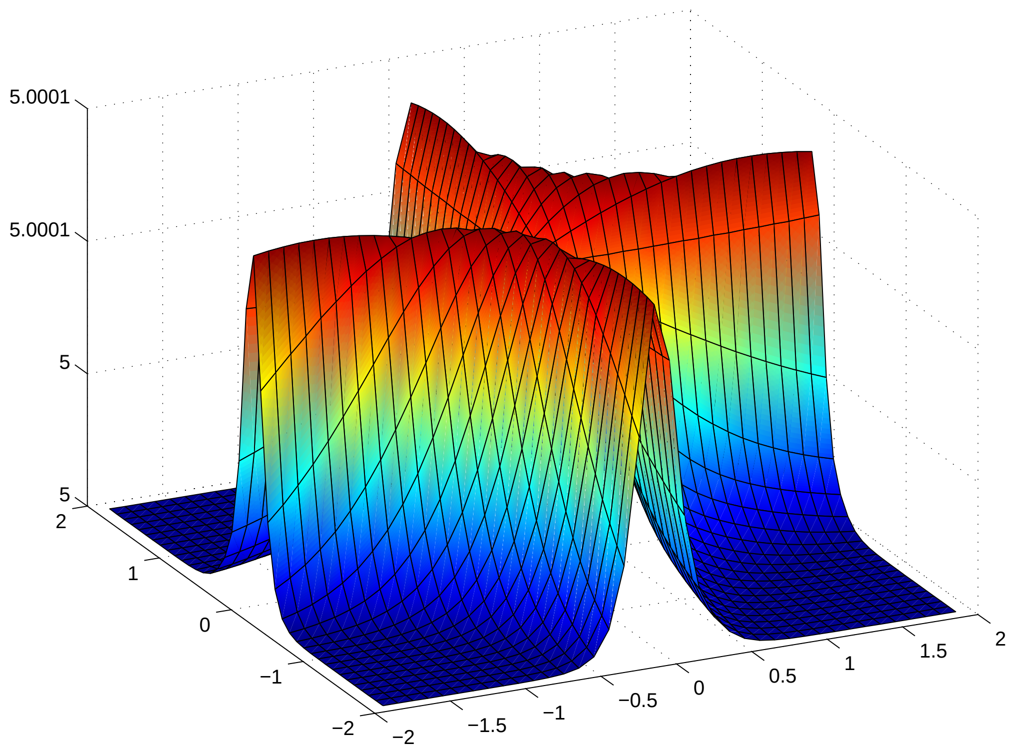
$$j_z\ (=\nabla^2\psi,\ 0\ \text{for}\ \psi = 2xy)$$



B_z > Poloidal B (B_{z0}=5)

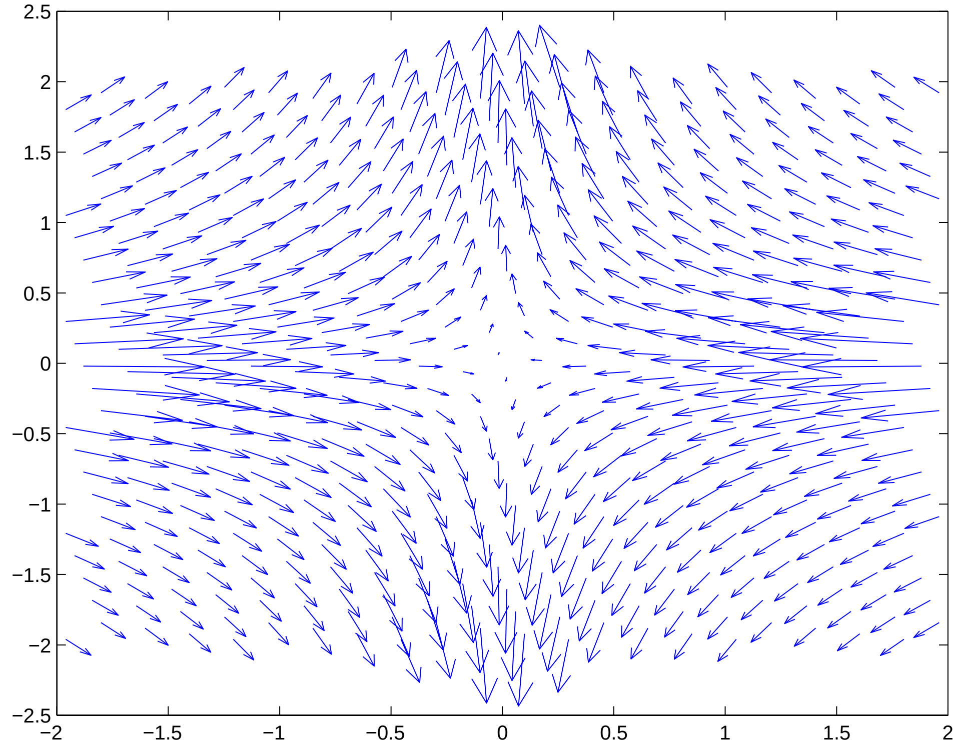
•Density, temperature, and poloidal B very close to B_{z0}=0 case

B_z: shifted by B_{z0}=5, no <B_{z0} ridges

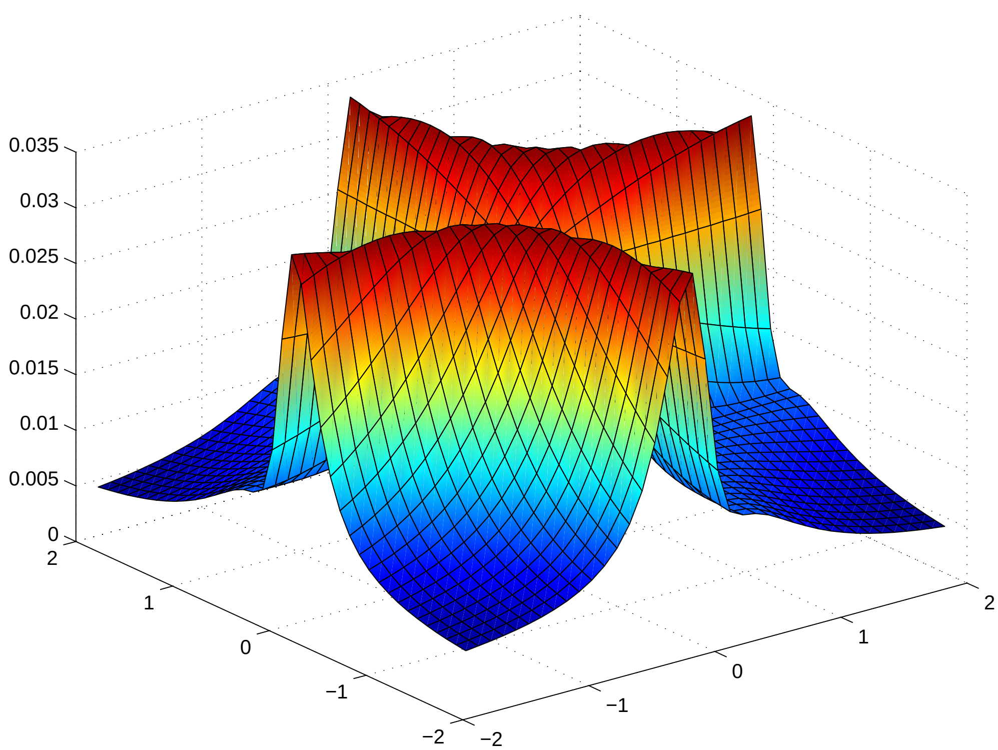


B_{z0}=5

Poloidal u

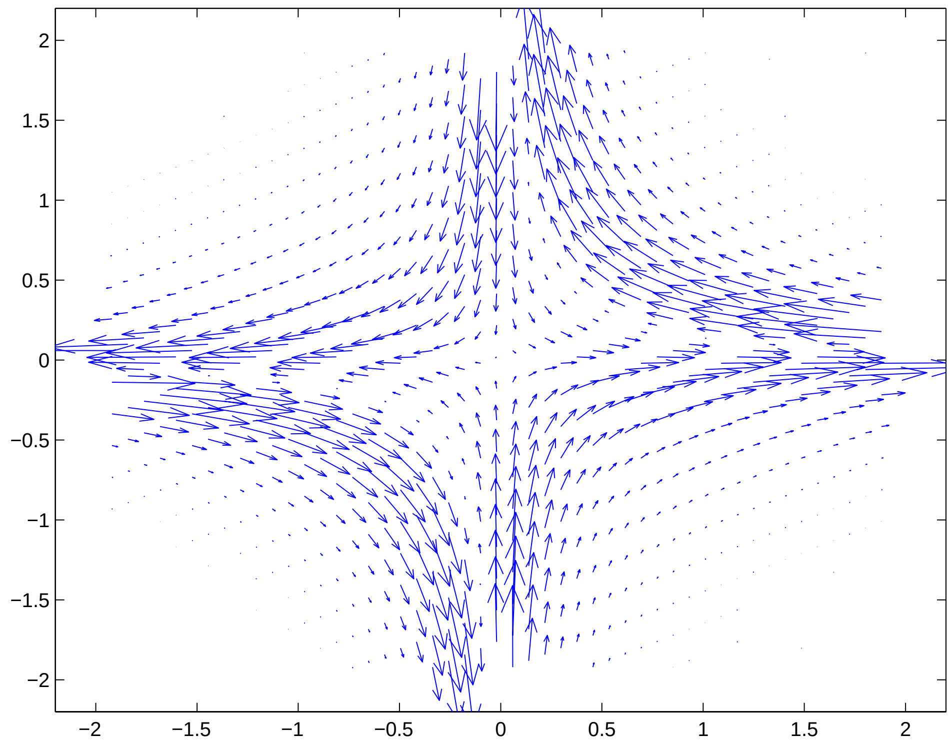


V_z (no negative ridges)

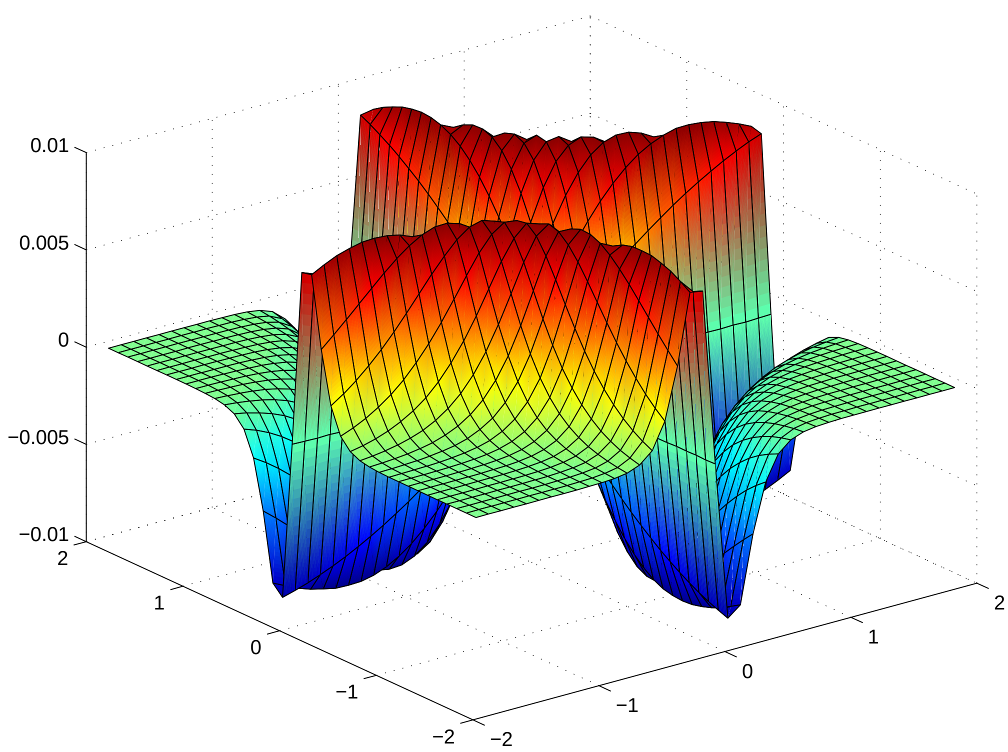


B_{z0}=5

Poloidal j (magnitude~1E-4)
guide field breaks symmetry among quadrants



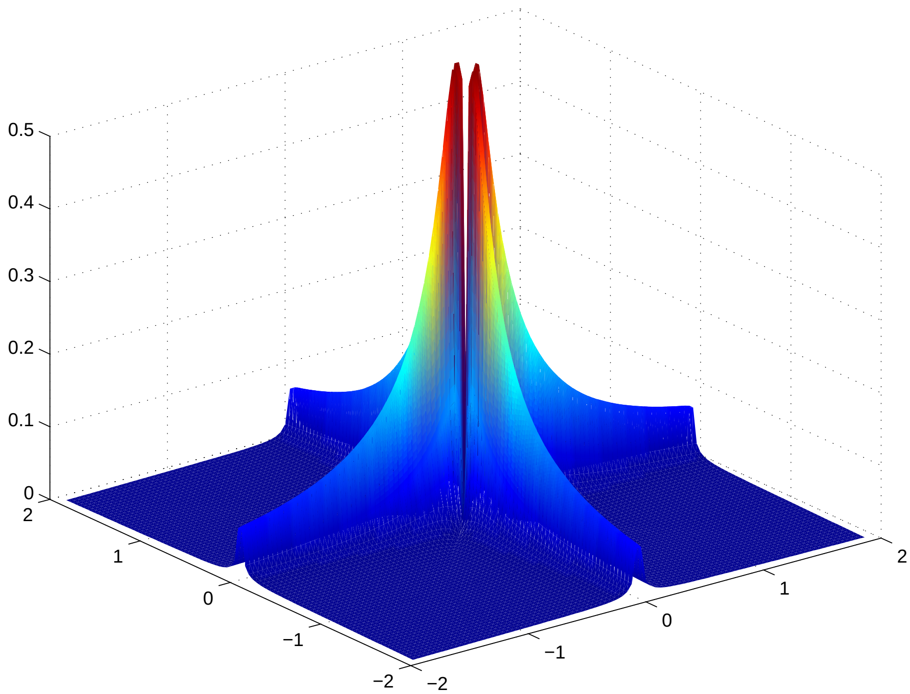
j_z



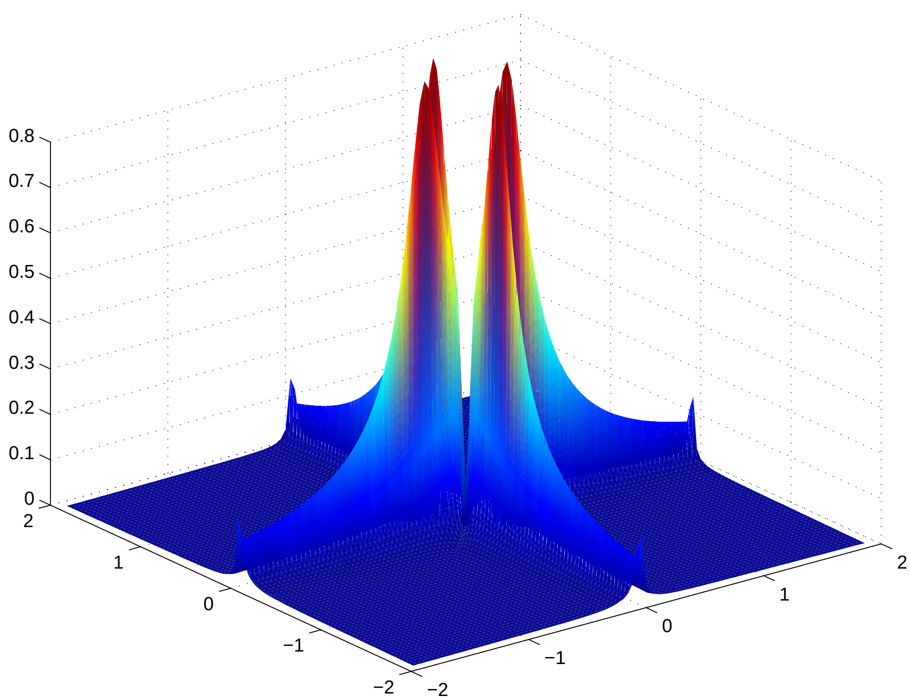
Possibility of Breaking Frozen-in Law

$$\frac{|\nabla\rho\times\nabla\psi|}{|\nabla\rho||\nabla\psi|}\neq 0$$

B_{z0}=0

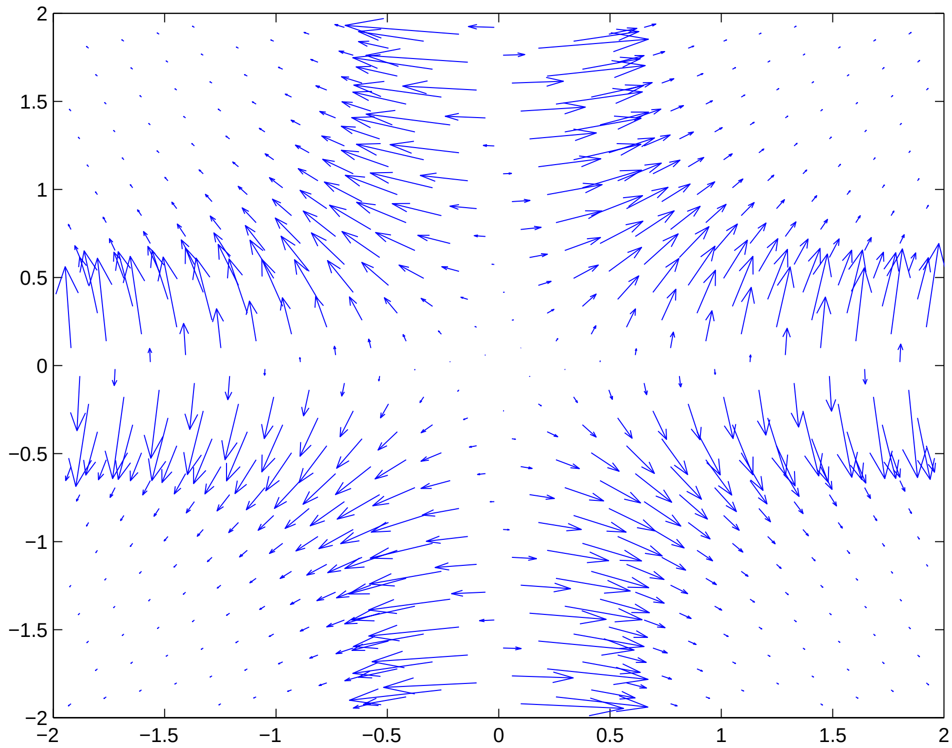


B_{z0} =5

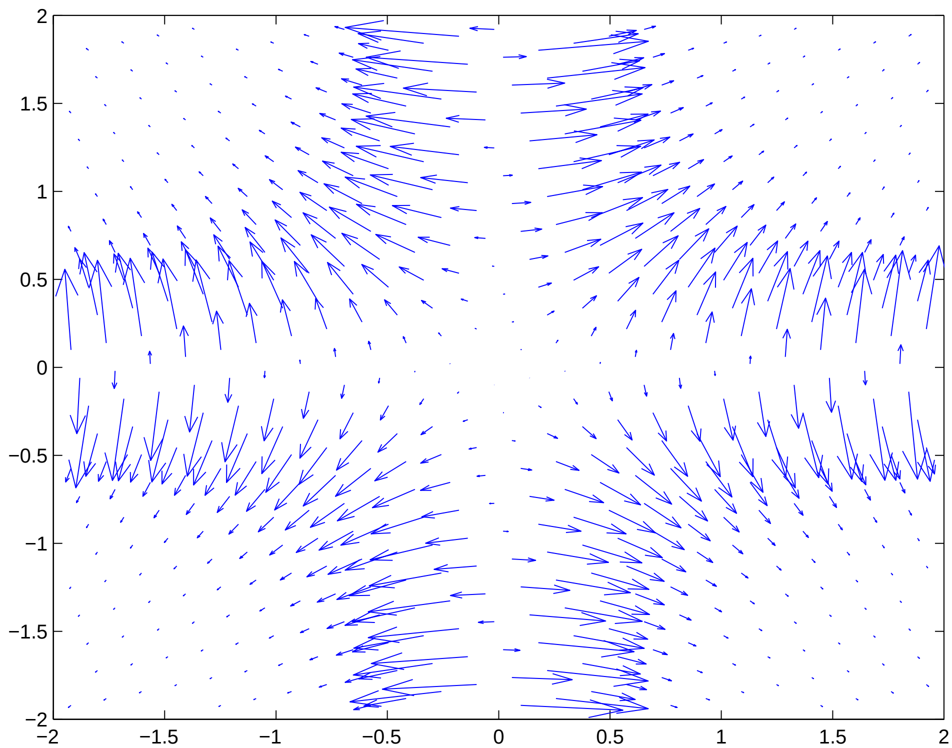


Electric Field From Ohm's Law

B_{z0}=0



B_{z0}=5



Relative Importance of terms, Integrated over domain

Poloidal Ohm

$$\vec{E}_p + \vec{u}\times\vec{B}_z + \vec{v}_z\times\vec{B}_p = \frac{1}{n}\left(\vec{j}_p\times\vec{B}_z + \vec{j}_z\times\vec{B}_p - \nabla p\right)$$

	$\vec{E}_p$	$\vec{u}\times\vec{B}_z$	$\vec{v}_z\times\vec{B}_p$	$\vec{j}_p\times\vec{B}_z$	$\vec{j}_z\times\vec{B}_p$	∇p
$B_{z0}=0$	0.64	7.3E-6	0.64	1E-7	1	1
$B_{z0}>B_p$	0.23	1	1	0.023	0.4	0.37

Poloidal Momentum

$$\vec{u}\cdot\nabla\vec{u} = \vec{j}_p\times\vec{B}_z + \vec{j}_z\times\vec{B}_p - \nabla p$$

	$\vec{u}\cdot\nabla\vec{u}$	$\vec{j}_p\times\vec{B}_z$	$\vec{j}_z\times\vec{B}_p$	∇p
$B_{z0}=0$	4.5E-3	1.6E-7	1	1
$B_{z0}>B_p$	4.6E-3	0.069	1	0.95

- z-comp. of Ohm: $\vec{u}\times\vec{B}_p = \vec{j}_p\times\vec{B}_p$, $E_z = 0$ up to roundoff
- z-comp. of Mom.: $\vec{u}\times\vec{B}_p = \vec{j}_p\times\vec{B}_p$ up to roundoff

Summary

• Hall-MHD equations simplified to 2 scalar PDEs
• Solutions exist with nontrivial plasma currents, flows, magnetic fields, and electric fields
• Possibility of breaking frozen-in law near separatrices since density not a flux function
•nonzero flows needed for breaking frozen-in

Future Prospects

•Time-dependent problem
•Applied electric field (as in VTF)
•Estimate of reconnection rate